and
$$\begin{tikzcd} S_{z^+} \arrow[r, "\Phi"] \arrow[rr, bend right, "u"'] & S_{z^-} \arrow[r, "u"] & M_\pm \end{tikzcd}$$

<u>Ex</u> $g = 2$

$\Gamma^+ = \{p_1^+\}$, $\Gamma^- = \{p_1^-, p_2^-\}$

$\Delta^{nd} = \{$two pairs$\}$

$\Delta^{br} = \{11$ pairs$\}$

$\Theta$ not shown

$\#\{$component of $S\} = 13$

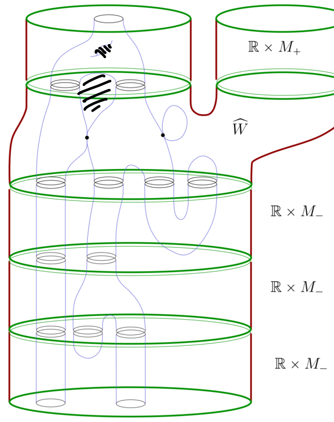
$N_+ = 1$

$0$

$-1$

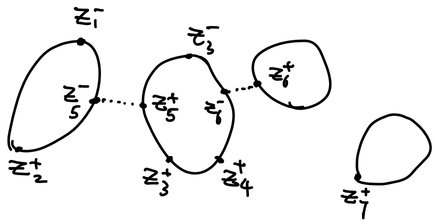
$-2$

$-N_- = -3$



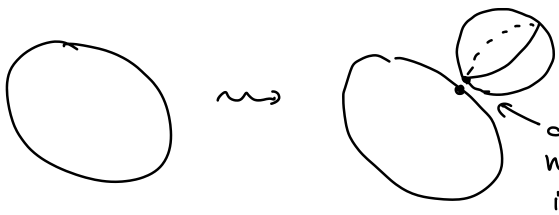

At level 0 (in $\widehat{W}$),

$$S\big|_{\text{level}=0} =$$




<u>Rmk</u> Given $(S, j, \Gamma \cup \Theta, \Delta, \Phi, L, u)$, one can modify in the following two way so that the images are more or less the same as $\mathrm{im}(u)$.

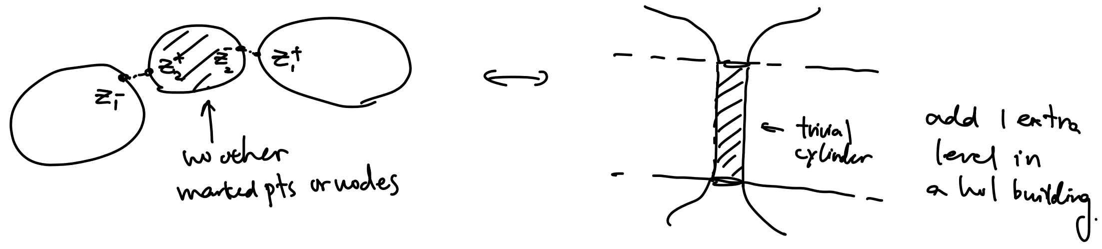
① 



add a new node in $\Delta^{nd}$

$\xrightarrow{u}$ require $u$ to be constant on this new $S^2$.

② add a new node in $\Delta^{br}$ in the following way:




To make the curve "unique", we require

- if $u$ is constant on component $\Sigma \subset \dot{S} \setminus \{\Gamma \cup \Theta \cup \Delta^{nd} \cup \Delta^{br}\}$, then $\chi(\Sigma) < 0$.
- no level in a hol building contains only trivial cylinders without any marked pts or nodes.

A hol building is called <u>stable</u> if it satisfies the two conditions above.

<u>Rmk</u> One can define equivalence of hol buildings (page 271 in [Wen]). The only difference is that in $\mathbb{R} \times M_\pm$, two maps are equ up to a shift along $\mathbb{R}$.

Denote

$$\overline{\mathcal{M}}_{g,l}(J, A, \gamma^+, \gamma^-) = \left\{ \begin{array}{l} \text{equ classes of stable hol buildings} \\ \text{in } (\hat{W}, \omega, J) \text{ with arith genus } g,\ l \text{ marked} \\ \text{pts, asymp to } \gamma^+, \gamma^-, \text{ and in class } A \end{array} \right\}$$

<u>Thm</u> (SFT compactness) Fix symp cobordism $(W, \omega_\varphi)$ with stable b/d and consider its completion $(\hat{W}, \omega_\varphi)$. Given a seq of a.c.s $J_k \to J$ in $\mathcal{J}((\omega_+, \lambda_+), (\omega_-, \lambda_-))$, $\gamma^+, \gamma^-$ sets of closed Reeb orbits on $(M_\pm, (\omega_\pm, \lambda_\pm))$, a seq of classes $A_k \in H_2(W, \gamma^+, \gamma^-)$. Then for any seq

$u_k : (\Sigma_k, j_k, \overset{\Gamma_k^+ \cup \Gamma_k^-}{\Gamma_k} \cup \Theta_k) \to (\hat{W}, \omega_\varphi, J_k)$ satisfying — asymp $\gamma^\pm$ — $im(u_k) = A_k$. — $J_k$-hol

if $E(u_k) < C$ for a uniform bound $C > 0$, then $\exists$ a stable hol

modified energy in $(\hat{W}, \omega_\varphi)$

building of height $N_- | 1 | N_+$ $(S, j, \Gamma \cup \Theta, \Delta^{nd}, \Delta^{br}, L, \Phi, u_\infty)$ s.t. when restricted

to a subseq, $u_k \to u_\infty$ in the follwng sense:

one can pass this to eqn classes.

When $k \gg 1$, $\exists$ homeomorphism $\varphi_k : \hat{S}_\Phi \simeq \Sigma_k$ where

$\hat{S}_\Phi$ is the compactification of $S \setminus (\Delta^{nb} \cup \Delta^{br})$, and $\varphi_k$ is

smooth outside $C_\Phi$ (circles from $\Delta^{nb} \cup \Delta^{br}$), mapping $\Gamma \cup \Theta$ to

$\Gamma_k \cup \Theta_k$ with order preserved, and $\varphi_k^* j_k \to j$ in $C^\infty_{l.c}(\hat{S}_\Phi \setminus C_\Phi)$.

(NEW)

Moreover for $N \in \{-N_-, \dots, 0, \dots, N\}$, let

$$V_k^N := u_k \circ \varphi_k \Big|_{\underbrace{S \setminus (\Gamma \cup \Delta^{nd} \cup \Delta^{br})}_{\subset \hat{S}_\Phi \setminus C_\Phi} \cap L^{-1}(N)} : S \setminus (\Gamma \cup \Delta^{nd} \cup \Delta^{br}) \cap L^{-1}(N) \to \hat{W}$$

$\leftarrow$ those components with level $N$

then

- $V_k^0 \to u_\infty^0 := u|_{0\text{-level components}}$ in $C^\infty_{l.c}(S \setminus (\Gamma \cup \Delta^{nd} \cup \Delta^{br}) \cap L^{-1}(0), \hat{W})$

- for each $|N| \geq 1$, $\exists\ r_k^N \to +\infty$ s.t.

$$\tau_{-r_k^N} \circ V_k^N \to u_\infty^N := u|_{N\text{-level components}} \text{ in } C^\infty_{l.c}(S \setminus (\Gamma \cup \Delta^{nd} \cup \Delta^{br}) \cap L^{-1}(N), \hat{W})$$

translation in $\mathbb{R}$-direction.

//.

Detailed proofs can be found in the following two standard refs:

Abbas: An intro to compactness in SFT. 2014 (Book)

BEHWZ: Compactness results in SFT. 2003 G&T.