Example $g=0$, for general $l \geq 3$.

For every nodal Riem surface $[(S, j, \Theta, \Delta)] \in \overline{M_{0,l}} \setminus M_{0,l}$,

$$2-l = 2-2g-l = \sum_{i=1}^{k} (2-2g_i - l_i - N_i)$$

l_i = # of marked pt in i-th component

N_i = # of nodal pt in i-th component

k = # component of S

- Total number of nodal pt $= \sum_{i=1}^{k} N_i/2$ and $\underbrace{k \leq 1 + \left(\sum_{i=1}^{k} N_i/2\right)}_{\text{b/c } S \text{ is connected.}}$
- $\sum_{i=1}^{k} l_i = l$

Then
$$2 = 2k - 2\sum_{i=1}^{k} g_i - \sum_{i=1}^{k} N_i$$
$$\Longleftrightarrow \quad 1 + \left(\sum_{i=1}^{k} N_i/2\right) = k - \sum_{i=1}^{k} g_i$$
$$\Longleftrightarrow \quad \sum_{i=1}^{k} g_i \leq 0 \Leftrightarrow g_i = 0 \quad \forall i = 1, \dots, k.$$

Hence, if we represent $[(S, j, \Theta, \Delta)]$ by a graph, it must be a tree.

Studying general top properties of $\overline{M_{g,l}}$ is a fundamental topic in many areas including Riem surface theory, GW, ...

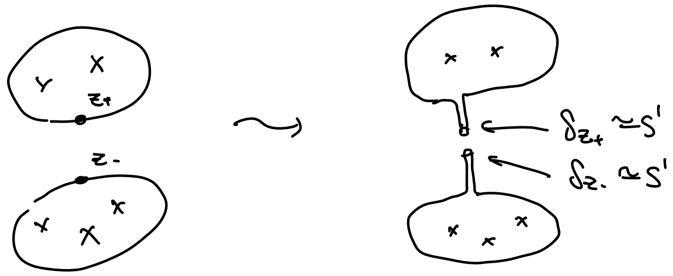
To confirm that $\overline{M_{g,l}}$ is (sequentially) compact in some sense, Let's introduce the following term.

For any punctured Riem surface $\dot{\Sigma} = \Sigma \setminus \Gamma$, near each punctured pt $p \in \Gamma$.

Denote $\overline{\Sigma}$ by replacing $[0,\infty)\times S^1$ by $[0,\infty]\times S^1$, then $\overline{\Sigma}$ is homeomorphic to a Riem surface with boundary and $\# \partial = \# \Gamma$.

Similarly, for (S, j, Θ, Δ), denote $\overline{S} := \overline{S \setminus \Delta}$.

e.g.



Then choosing a orientation preserving map $\Phi : \delta_{z_+} \to \delta_{z_-}$, we can glue z_+, z_- together and get $\hat{S}_\Phi := \overline{S}/\sim$.

One should view $\hat{S}_\Phi$ as a smoothing of $\hat{S}$ (with nodal singularities).
depending on Φ.

Rmk For convenience, denote the gluing circle (by $\delta_{z_+} \overset{\Phi}{\simeq} \delta_{z_-}$) by $C_\Phi \subset \hat{S}_\Phi$.

Thm (Thm 9.26 in [Wen]) For g, l satisfying $2g + l \geq 3$. Then for any seq $[(\Sigma_k, j_k, \Theta_k)] \in M_{g,l}$, there exists a stable connected nodal Riem surface $[(S, j, \Theta, \Delta)] \in \overline{M_{g,\ell}}$ s.t. after restricting to a subseq,

$$[(\Sigma_k, j_k, \Theta_k)] \longrightarrow [(S, j, \Theta, \Delta)]$$

in the sense: (S, j, Θ, Δ) admits a "decoration" Φ s.t. when $k \gg 1$

$\exists$ homeomorphism $\varphi_k : \hat{S}_\Phi \simeq \Sigma_{k_\Lambda}$ mapping Θ to Θ_k preserving the order, $\varphi_k^* j_k \to j$ in $C^\infty_{loc}(\hat{S}_\Phi \setminus C_\Phi)$.

smooth outside C_Φ.

In particular when $k \gg 1$, Σ_k are in fixed type Σ.

Θ_k are in the same number.

<u>Rmk</u> $M_{g,\ell}$, $\overline{M_{g,\ell}}$ are about <u>domains</u>, NOT curves or maps.

One can fix a seq of a.c.s in the target (M, ω), J_k $(\to J)$ and consider

$$u_k : (\Sigma_k, j_k, \Theta_k) \longrightarrow (M, \omega, J_k)$$

which are J_k-hol maps. This naturally lead to the notation when (M, ω) is a closed symp mfd,

$$M_{g,\ell}(J, A) = \Big\{ \Big(\underbrace{u : (\Sigma, j) \to (M, J)}_{J\text{-hol and } [\mathrm{im}(u)] = A} \Big) \times (\Sigma^\ell \setminus \Delta) \Big\} \Big/ \underset{\text{biho/+ order preserving}}{\sim}$$

where $A \in H_2(M)$

Then $[u_k] \in M_{g,\ell}(J_k, A_k)$ with uniform energy upper bound $\xRightarrow[\text{compactness}]{\text{Gromov}}$ $u_\infty : (S, j, \Theta, \Delta) \to (M, \omega, J)$

u_∞ is J-hol + $[u_\infty] = A$

domain is $\hat{S}_\Phi$ for some Φ.

Difference between this version of Gromov compactness with the one in previous lecture without marked pts:

$\hat{S}_{\Phi}$ itself admits nodal pts in the DM-compactification that do not come from bubbling! ← These nodal pts come from coincidence of marked pts.

Rmk - Recall the domain of the previous Gromov compactness is called "bubble tree" ← (WHY? :))

- Bubbles in this Gromov compactness sit in $\hat{S}_{\Phi} \setminus C_{\Phi}$.

2. SFT compactness

Setting: symplectic cobordism (W, ω) and its completion $\hat{W}$.

The analogue of $M_{g,l}(J, A)$ is

$$M_{g,l}(J, A, \gamma^+, \gamma^-)$$

where $\gamma^+ = \{\gamma_1^+, \dots, \gamma_{k_+}^+\}$ ← each is a closed Reeb orbit of the positive end of W, $(M_+, (\omega_+, \lambda_+))$

$\gamma^- = \{\gamma_1^-, \dots, \gamma_{k_-}^+\}$ ← ...

$A \in H_2(W; \gamma^+, \gamma^-)$ and J is a compatible a.c.s on $(\hat{W}, \omega)$.

$\exists\ \gamma^+, \gamma^- \Rightarrow$ we need to modify the definition of nodal Riem surface to be "punctured" nodal Riem surface

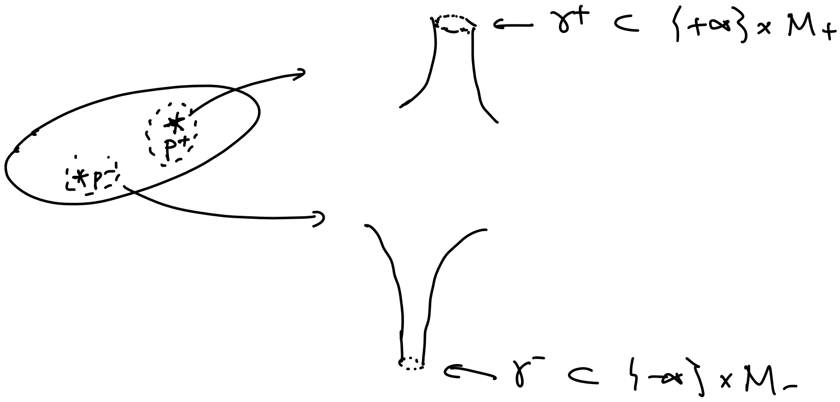
$(S, j, \Gamma \cup \Theta, \Delta)$

NEW!

$\Gamma = \{\underbrace{p_1^+, \dots, p_{k_+}^+}_{\text{collection is denoted by } \Gamma^+}, \underbrace{p_1^-, \dots, p_{k_-}^-}_{\text{collection is denoted by } \Gamma^-}\}$

Elements in Γ are also called marked pts, and we will conside $\dot{S} = S \setminus \Gamma$ when put into curve setting.

e.g.



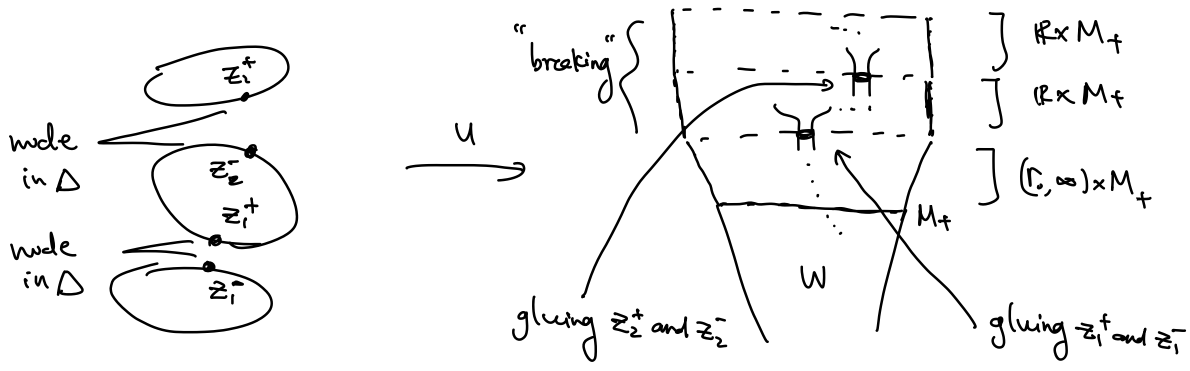
- $(S_0, j_0, \Gamma_0 \cup \Theta_0, \Delta_0) \sim (S_1, j_1, \Gamma_1 \cup \Theta_1, \Delta_1)$ same as above
- connected, stable, arithmetic genus same as above.

Let's consider curves or maps:

$$u: (S, j, \Gamma \cup \Theta, \Delta) \longrightarrow (\hat{W}, \omega)$$

Here is a NEW phenomenon (in the image), due to non-cpt of $\hat{W}$.

symplectization part of $\hat{W}$: $[r, \infty) \times M_+ \simeq [r_0, \infty) \times M_+ \cup \underbrace{\mathbb{R} \times M_+ \cup \cdots}_{\text{may have many copies.}}$

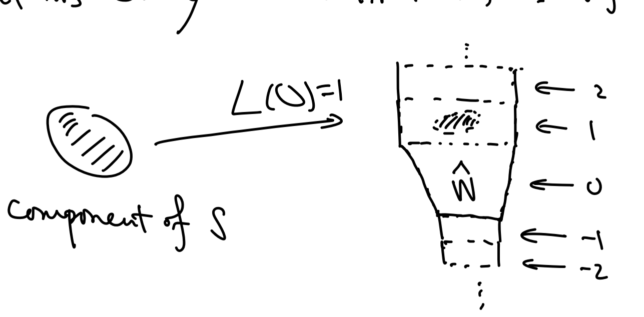


These nodes are not the same as the standard node as above. To describe this more rigorously, we will need the following definition.

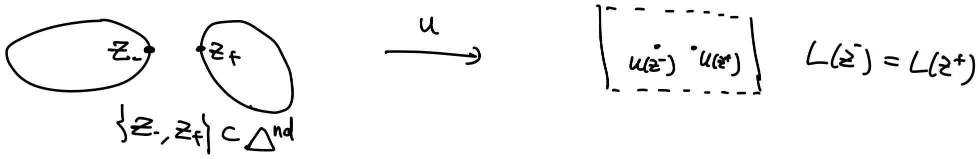
Def Given $g, l, N_-, N_+ \geq 0$, a holomorphic building of height $N_-|1|N_+$ with arith genus g and l marked pts is a tuple

$$\Big(S, j, \overset{\Gamma^+\cup\Gamma^-}{\overset{\shortparallel}{\Gamma}} \cup \Theta, \Delta = \underset{\substack{\uparrow \\ \text{standard nodes}}}{\Delta^{nd}} \cup \underset{\substack{\nwarrow \\ \text{breaking nodes}}}{\Delta^{br}}, L, \Phi, u\Big)$$

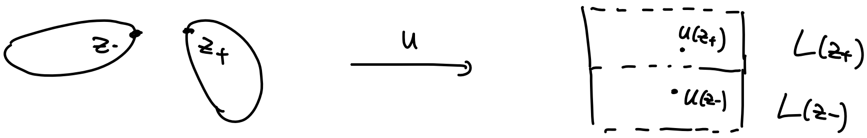
- $L : S \longrightarrow \{-N_-, \cdots, -1, 0, 1, \cdots, N_+\}$ locally constant map (called the level structure) s.t.
  - L attains every value in $\{-N_-, \cdots, N_+\}$ except possibly 0; this labels the height level and $\hat{W}$ is labelled by 0.



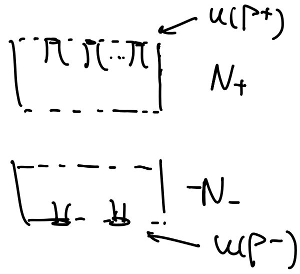
- for node $\{z^-, z^+\} \subset \Delta^{nd}$, $L(z^-) = L(z^+)$.



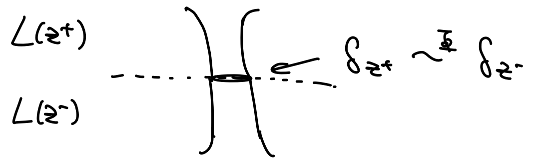
- for breaking node $\{z^-, z^+\} \subset \Delta^{br}$, $L(z^+) - L(z^-) = 1$



- $L(\Gamma^+) = \{N_+\}$ and $L(\Gamma^-) = \{-N_-\}$



- Φ is the "decoration" is a set of choice of orientation-reversing map $\delta_{z^+} \xrightarrow{\Phi} \delta_{z^-}$ for breaking node $\{z^+, z^-\} \subset \Delta^{br}$



- $u : (S \setminus (\Gamma \cup \Delta^{br}), j) \longrightarrow \coprod_{N \in \{-N_-, \cdots, N_+\}} (\hat{W}_N, J_N)$

where
$$(\hat{W}_N, J_N) = \begin{cases} (\mathbb{R} \times M_+, J_+) & \text{for } N \in \{1, \cdots N_+\} \quad \text{upper levels} \\ (\hat{W}, J) & \text{for } N = 0 \\ (\mathbb{R} \times M_-, J_-) & \text{for } N \in \{-N_-, \cdots, -1\} \quad \text{lower levels} \end{cases}$$