# 1. Deligne – Mumford compactification

$(\Sigma_g, j)$ : closed Riem surface with genus $g$.

$\Theta = \Theta(p_1, \cdots, p_\ell)$ : an ordered set of $\ell$-many pts on $\Sigma_g$

distinct

marked pts and view it as $\ell$-tuple $\subset \Sigma_g^\ell \setminus \Delta$

$(\Sigma_g, j, \Theta)$ called a pointed Riem surface

Define $(\Sigma_{g_1}, j_1, \Theta_1) \sim (\Sigma_{g_2}, j_2, \Theta_2)$ iff $\exists$ a biholomorphism $\varphi$ from $(\Sigma_{g_1}, j_1)$ to $(\Sigma_{g_2}, j_2)$ (so $g_1 = g_2$) and $\varphi(\Theta_1) = \Theta_2$ with respect to ordering.

this already fixes $\Sigma_g$ topologically

$$\Rightarrow \quad M_{g,\ell} = \{(\Sigma_g, j, \Theta)\} / \sim$$

# of marked pts

Rmk One can equip some geometric str on $M_{g,\ell}$.

Recall a groupoid is a cat where all morphisms are isomorphisms

invertible

i.e. space of orbits

Then Grothendieck says any moduli = an orbit space under the action of a groupoid.

Rewrite $\mathcal{J}(\Sigma_g) = \{$all cpx strs on $\Sigma_g\}$

$\mathrm{Diff}(\Sigma_g) = \{$all diffeos on $\Sigma_g\}$ $\Rightarrow$ $\forall \varphi \in \mathrm{Diff}(\Sigma_g)$ $\varphi^* j$ defines an element in $\mathcal{J}(\Sigma_g)$

$(\varphi^* j)_x := (d\varphi)_x^{-1} \cdot j_{\varphi(x)} \cdot (d\varphi)_x$

$\overset{\shortparallel}{(d\varphi^{-1})_{\varphi(x)}}$

Then to take care of the flexibility of $j$ in $(\Sigma_g, j)$, one

Can consider quotient space $\mathcal{J}(\Sigma_g)/\mathrm{Diff}(\Sigma_g)$ (usually denoted by $M_g$, the moduli space of cpx str on $\Sigma_g$)

This confirms what Grothendieck's statement above

$$\text{Quotient} \simeq \left\{ \begin{array}{ll} \text{orbit 1} & j_1 \to \varphi^* j_1 \to \cdots \\ \text{orbit 2} & j_2 \to \varphi^* j_2 \to \cdots \\ \cdots & \end{array} \right.$$

In the same way, one can add ordered pts $\Theta$ and consider

$$\left(\mathcal{J}(\Sigma_g) \times (\Sigma_g^{\ell} \setminus \Delta)\right) / \mathrm{Diff}(\Sigma_g) \leftarrow \text{acting by diagonal.}$$

- It is $\simeq \mathcal{J}(\Sigma_g) / \mathrm{Diff}(\Sigma_g, \Theta)$ for any fixed $\Theta$.

  where $\mathrm{Diff}(\Sigma_g, \Theta) \leq \mathrm{Diff}(\Sigma_g)$ that fixes $\Theta$ (stabilizer of $\mathrm{Diff}(\Sigma_g)$ w.r.t $\Theta$).

- It can also be identified with $M_{g,\ell}$: $(\Sigma_g, j) \overset{\varphi}{\simeq} (\Sigma_g, j')$ iff

  $d\varphi \cdot j = j' \cdot d\varphi$ iff $j' = d\varphi \cdot j \cdot (d\varphi)^{-1}$.

In other words,

$$M_{g,\ell} \simeq \left(\mathcal{J}(\Sigma_g) \times (\Sigma_g^{\ell} \setminus \Delta)\right) / \mathrm{Diff}(\Sigma_g) \simeq \mathcal{J}(\Sigma_g) / \mathrm{Diff}(\Sigma_g, \Theta)$$

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Introducing notation $\chi(\Sigma_g \setminus \Theta)$ = Euler char of punctured $\Sigma_g$

$= 2 - 2g - \ell$.

<u>FACT</u> (Prop 7.9 in [Wen]) When $\chi(\Sigma_g \setminus \Theta) < 0$, for each $j \in \mathcal{J}(\Sigma_g)$

It is a Lie group as another fact $\rightarrow$ $\mathrm{Aut}(\Sigma, j, \Theta)$ = stabilizer of $\mathrm{Diff}(\Sigma_g, \Theta)$ w.r.t $j$ is finite