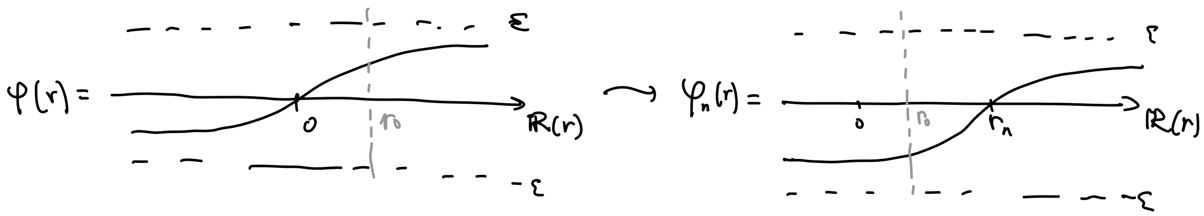
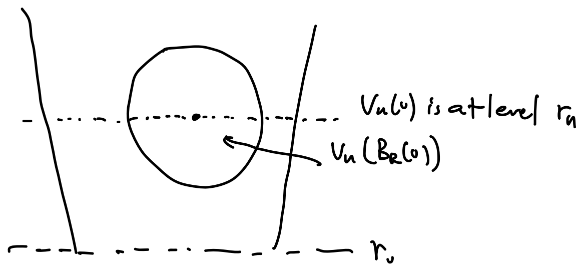
where $\varphi_n(r) = \varphi(r - r_n)$.



For this R, when $n \gg 1$, $B_R(0) \subset B_{R_n}(0)$, so

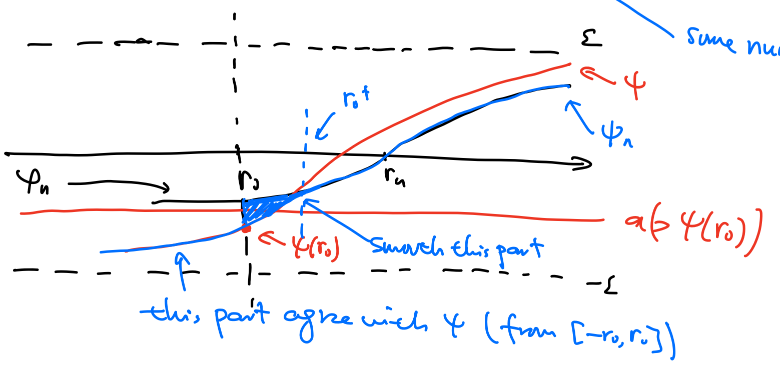
$$V_n(B_R(0)) \subset V_n(B_{R_n}(0)) \subset [r_0, \infty) \times M_+ .$$



$\Rightarrow$ for integration in $(*)$ we only need information of $\varphi_n(r)$ for $r \geq r_0$

$\Rightarrow$ for each n, $\exists\, \psi_n \in T_{\varphi,\varepsilon,r_0}$ s.t. $\psi_n\big|_{[r_0^+,\infty)} = \varphi_n\big|_{[r_0^+,\infty)}$

some number $r_0^+ > r_0$



Take ψ_n = an extension of $\varphi_n\big|_{[r_0^+,\infty)}$ (s.t. that it agrees with φ in $[-r_0, r_0]$
$\Rightarrow \psi_n \in T_{\varphi,\varepsilon,r_0}$)

Rmk One can take smoothing in the "blue" region so r_0^+ can be taken arbitrarily close to r_0.

then

$$\int_{B_R(0)} v_n^*(\omega_+ + d(\varphi_n(r)\lambda_+)) = \int_{B_R(0)} v_n^*(\omega_+ + d(\psi_n(r)\lambda_+))$$

$$\leq \int_{B_{\varepsilon_n |du(w_n)|}(0)} v_n^*(\omega_+ + d(\psi_n(r)\lambda_+))$$

$$= \int_{B_{\varepsilon_n}(w_n)} u^*(\omega_+ + d(\psi_n(r)\lambda_+))$$

$$\leq \int_{\mathcal{U}} u^*(\omega_+ + d(\underbrace{\psi_n(r)}_{\in T_{\psi,\varepsilon,r_0}}\lambda_+)$$

$$\leq E_{\psi,\varepsilon,r_0}(u|_{\mathcal{U}}) \quad (< +\infty).$$

This holds for any R (the upper bound is independent of R), so we get the conclusion in claim $\square$

This claim in particular implies that

$$E_\varepsilon(u|_{\mathcal{U}}) = \sup_{\varphi \in T_\varepsilon} \int_{\mathcal{U}} u^*(\omega_+ + d(\varphi(r)\lambda_+) < +\infty.$$

(recall that boundedness of E_ε for $\mathbb{R} \times M_+$ is ind of the target of φ: either $(-\varepsilon,\varepsilon)$ or any interval $(a,b) \subset (-\varepsilon,\varepsilon)$).

Moreover, we claim that $\int_{\mathbb{C}} v_\infty^* \omega_+ = 0$.

Pf of claim. Take subseq of uec. to consider disjoint NBH of w_n

Then obviously $\lim_{n\to\infty} \int_{B_{\varepsilon_n}(w_n)} u^* \omega_\varphi = 0$ for any $\varphi \in T_{\psi,\varepsilon,r_0}$.

$$\underset{\text{fixed } R>0}{\overset{\text{for any}}{\Rightarrow}} \quad 0 = \lim_{n\to\infty} \int_{B_{\varepsilon_n |du(w_n)|}(0)} V_n^* \omega_\varphi$$

$\tilde{V}_n = \tau(-r_n)\cdot V_n$

$$= \lim_{n\to\infty} \int_{B_{\varepsilon_n |du(w_n)|}(0)} \tilde{V}_n^* \cdot \tau(r_n)^* \omega_\varphi$$

$$\geq \lim_{n\to\infty} \int_{B_R(0)} \tilde{V}_n^* \cdot \tau(r_n)^* \omega_\varphi = \lim_{n\to\infty} \int_{B_R(0)} \tilde{V}_n^* (\omega_+ + d(\varphi_n(r)\lambda_+))$$

where $\varphi_n(r) = \varphi(r + r_n)$.

Note that $\omega_+ + d(\varphi_n(r)\lambda_+) = \omega_+ + \varphi_n(r) d\lambda_+ + \varphi_n'(r) dr \wedge \lambda_+$.

Take $\varphi \in T_{\psi,\varepsilon,r_0}$ s.t. $\varphi'(r) \to 0$ as $r \to +\infty$. Then

$$\varphi_n'(r) = \varphi'(r + r_n) \to 0 \quad \text{as } n \to +\infty.$$

$$\underset{\substack{\text{computation}\\ \text{above}}}{\overset{\text{cont.}}{\Rightarrow}} \quad 0 \geq \lim_{n\to\infty} \int_{B_R(0)} \tilde{V}_n^* (\omega_+ + \varphi_n(r) d\lambda_+) \qquad (**)$$

Since ω_+ is non-deg on the $\ker\lambda_+$, when ε is sufficiently small, $\lim_{n\to\infty} \varphi_n(r) = \varphi(+\infty) \leq \varepsilon$, so $\omega_+ + \varphi_n(r) d\lambda_+$ is also non-deg on $\ker\lambda_+$.

Hence, $(**)$ implies $V_\infty(\mathbb{C})$ is ~~everywhere~~ on $B_R(0)$ for any R tangent to ∂_r and $R_{(\omega_+,\lambda_+)}$

$\Rightarrow \int_{\mathbb{C}} V_\infty^* \omega_+ = 0$ the claim's conclusion □

Then by energy control estimation, (beginning of SFT-7) two claims above implies that $V_\infty =$ constant. $\rightarrow\leftarrow$.

<u>Case 3</u> $V_k(0)$ diverge to $\{-\infty\} \times M_-$

A symmetric argument as in Case 2. □.

For a fixed $u: U$ (nbh of puncture z of Σ) $\to \hat{W}$ with bounded energy, one gets a seq of J-hol curve by "sliding technique": consider the identification

$U = \mathbb{D}^2$

this step does NOT change the boundedness of the energy

(similarly to $(-\infty,0]\times S^1$)

Then consider a seq $s_n \to +\infty$ and $u_n : [-s_n,\infty)\times S^1 \to \hat{W}$ by

$$u_n(s,t) := u(s+s_n, t).$$

strictly speaking, this should be u composed with the bihol identification.

By Prop above, $|du_n|$ has a uniform upper bound (C^1-bound), but it's not clear or not even true that $\mathrm{im}(u_n)$ is uniformly bounded (C^0-bound).

There are three cases:

① $u_n(0,0) = u(s_n, 0)$ has a bounded seq.

$\overset{+\text{Prop}}{\Longrightarrow}$ $\exists$ subseq $u_n \to$ a J-hol $u_\infty : \mathbb{R}\times S^1 \to \hat{W}$.

For any $\varphi \in T_{\psi,\varepsilon,r_0}$ and any fixed $R>0$, we have

$$\int_{[-R,R]\times S^1} u_\infty^* \omega_\varphi = \lim_{n\to\infty} \int_{[-R,R]\times S^1} u_n^* \omega_\varphi$$

$$\leq \lim_{n\to\infty} \int_{[-\frac{s_n}{2},\infty)\times S^1} u_n^* \omega_\varphi$$

$$= \lim_{n\to\infty} \int_{\underbrace{[s_n/2,\infty)\times S^1}_{\to\{z\}}} u^*\omega_\varphi = 0$$

(b/c by our hypothesis near $\{z\}$ the energy $E_{\varphi,\varepsilon,r_0}$ is bounded.)

$$\overset{\text{Let } R\to+\infty}{\Longrightarrow} \int_{\mathbb{R}\times S^1} u_\infty^*\omega_\varphi = 0 \quad (\forall \varphi\in T_{\varphi,\varepsilon,r_0})$$

$\Longrightarrow$ u_∞ is constant mapping $\mathbb{R}\times S^1$ to a pt $p\in\widehat{W}$.

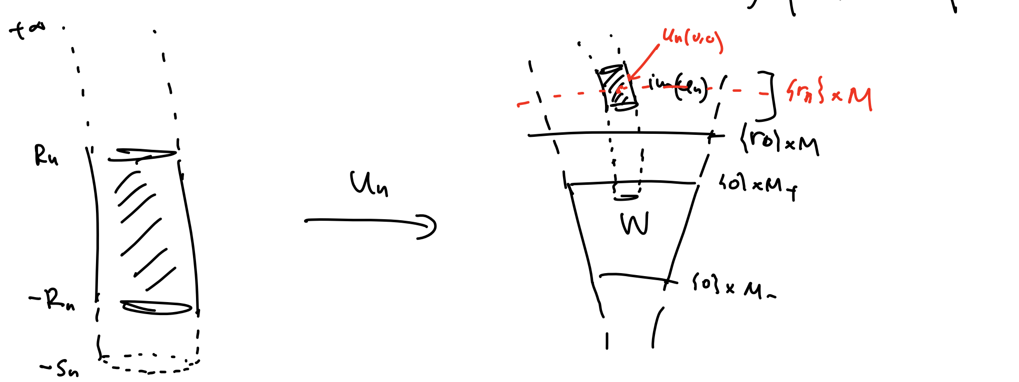
$\Longleftrightarrow$ $u_n(0,\cdot) = u(s_n,\cdot) \to p$ when $n\to+\infty$.

② $u_n(0,0) = u(s_n,0)$ has a subseq $\to \{+\infty\}\times M_+$.

Again, assume $u_n(0,0)\in\{r_n\}\times M_+$ when $r_n\to+\infty$.

$\exists$ uniform upper bound of $|du_n|$ $\Longrightarrow$ $\forall n$, $\exists$ $R_n \to +\infty$ s.t. REF.

$$u_n\big([-R_n,R_n]\ (\subset[-s_n,+\infty))\big) \subset \underbrace{[r_0,\infty)}_{\text{the symplectization part}}\times M_+$$



Now, let's slide the cylinder

$$\tilde{u}_n := \tau(-r_n)\circ u_n\big|_{[-R_n,R_n]\times S^1} : [-R_n,R_n]\times S^1 \to \mathbb{R}\times M_+$$

Then $|d\tilde{u}_n|$ uniformly bounded and also $\{\tilde{u}_n(0)\}$ is a bounded seq.

by case ①, we know

$$\exists\ \tilde{u}_n \longrightarrow u_\infty : \mathbb{R}\times S^1 \longrightarrow \mathbb{R}\times M_+ .$$

which is J_+-hol (= extension of $J|_{[r_0,\infty)\times M_+}$).

By the same argument as in the previous prop, we get

$$E_\Sigma(u_\infty) < +\infty \quad \text{and} \quad \int_{\mathbb{R}\times S^1} u_\infty^* \omega_+ = 0$$

$\Longrightarrow$ by energy control prop (beginning of SFT-7)

u_∞ is either a constant or u_∞ is a reparametrization of "trivial cylinder" $u_\gamma : \mathbb{R}\times S^1 \to \mathbb{R}\times M_+$ by $(s,t) \to (Ts, \gamma(Tt))$

for some closed Reeb orbit of $(M_+, (\omega_+,\lambda_+))$ of period T.

③ $u_n(0,0) = u(s_n, 0)$ has a subseq $\longrightarrow \{-\infty\}\times M_-$.

The same discuss and result as in case ②.

Rmk For puncture pt z of Σ:

- If u is bounded, then removal of singularities can apply, so under the bounded energy condition, z is removable

applicable when the image of u is contained in cpt subset.

- If u is unbounded, then Lemma 9.16 in [Wen] proves that u is proper: $\forall R \geq r_0$, $\exists s_0 \geq 0$ s.t.

$$u((s_0,\infty)\times S^1) \subset (R,\infty)\times M_+$$
$$\text{or } \subset (-\infty,-R)\times M_-$$

in other words, mapping nbh of puncture z to nbh of $\{+\infty\} \times M_+$ or $\{-\infty\} \times M_-$.

Here is a summary, under (uniform) bounded energy control:

target / domain	compact (or contained in cpt subset)	non-compact (completion of a symp cob)
$z \in \Sigma$ cpt	u: regular pt	constant map or reduce back to the left case ← due to maximum principle.
	$\{u_n\}$: bubbles when $\|du_n\| \to +\infty$.	
puncture point z of $\dot{\Sigma}$ non-cpt	removable singularity	asymptotic to closed Reeb orbits.
	$\{u_n\}$: each u_n reduces back to case above + bubble	$\{u_n\} \longrightarrow$?? — asymp end / bubble / breaking (cf. Ham Floer cylinder)

To rigorously describe this limit (usually called a "holomorphic building") we need to build up some basic language and notations (see Next Lecture).