Rmk One more thing needs to be confirmed / defined: if

$$u_n \longrightarrow u^{(1)} \# u^{(2)} \# \cdots \# u^{(m)}$$

$$u_n \longrightarrow \tilde{u}^{(1)} \# \tilde{u}^{(2)} \# \cdots \# \tilde{u}^{(m')}$$

then $m = m'$ and $\tilde{u}^{(i)}$ = a shift of $u^{(i)}$.

Back to the Ham Floer homology setting, we have a similar result as the Cor near the end of previous section.

Cor. For x_- and x_+, closed Ham orbits of the Ham sys (M, ω, H, J) for every $c > 0$, $\exists$ only finitely many htp classes $A \in \pi_2(M; x_\pm)$ that can be represented by a Floer cylinder $u: \mathbb{R} \times S^1 \to M$ with $E(u) < c$.

Pf Suppose not, $\exists$ a seq of Floer cylinders with uniform bd-energy

$$u_n : (\mathbb{R} \times S^1, j) \to (M, \omega, J) \text{ with } [\mathrm{im}(u_n)] = A_n \text{ and}$$

A_n are all different.

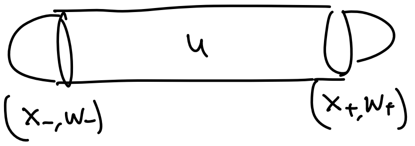
$\xRightarrow{\text{by Thm}}$ $\exists$ a subseq (still denoted by) $u_n \longrightarrow u$ (broken + bubble).

when $n \gg 1$, $[\mathrm{im}(u_n)]$ is stable. $\rightarrow\leftarrow$ $\square$

Rmk - Consider "capped" closed Ham orbits $(x_\pm, w_\pm)$ where

$$w_\pm : \mathbb{D}^2 \to M \text{ with } w_\pm|_{\partial \mathbb{D}^2} = x_\pm .$$

Then Floer cylinders $u \in \mathcal{M}((x_-, w_-), (x_+, w_+))$ counted in Ham Floer homology theory satisfies



with $w_+ = w_1 \# u$

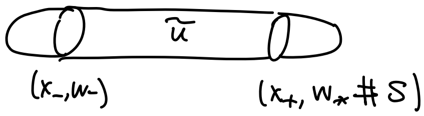
Then $E(u) = A_H((x_-,w_-)) - A_H((x_+,w_+)) < C$ ← ind of u

$\overset{\text{Cor}}{\Longrightarrow} \{[\mathrm{im}(u)]\}_{u\in M((x_-,w_-),(x_+,w_+))}$ is a finite set in $\pi_2(M; x_\pm)$

Note that this is a highly non-trivial result!

- Sometimes we view this from a different perspective: given (x_-,w_-) and $(x_+, \underset{\text{fixed}}{w_*})$, then consider Floer cylinder u starting from (x_-,w_-) and connecting x_+.
  (capping of x_+ is determined by u)

(so $w_* = w_- \# u$ for some u)



(so $w_* \# S = w_- \# \tilde{u}$ for some other u, where S is a top 2-sphere).

To facilitate the encoding of information on "S", one introduce the notation $T^{[S]}$, s.t.

Here, we can either view S as a homotopy class and then use the second Hurewicz map to map S into H_2(M; \Z) or we can directly view S as a homology class. Both are OK.

$$(x_-,w_-) \longrightarrow (x_+,w_*) \qquad (x_-,w_-) \longrightarrow (x_+,w_*)\cdot T^{[S]}$$

← homotopy class $\pi_2(M)$

$$\xrightarrow[\text{all such } u]{\text{exhaust}} (x_-,w_-) \longrightarrow (x_+,w_*)\cdot \underbrace{(a_{S_1}T^{S_1} + a_{S}T^{S_2}+\cdots)}$$

Claim: it's an element in Novikov ring, in the sense that $[\omega](S_n) \to \underset{\text{by convention}}{\underline{\underline{+\infty}}}$ as $n\to\infty$.

b/c for any $C\in\mathbb{R}$, if $[\omega](S) < C$, then any corresponding u satisfies

$$E(u) = A_H((x_-,w_-)) - A_H((x_+, w_*\# S))$$

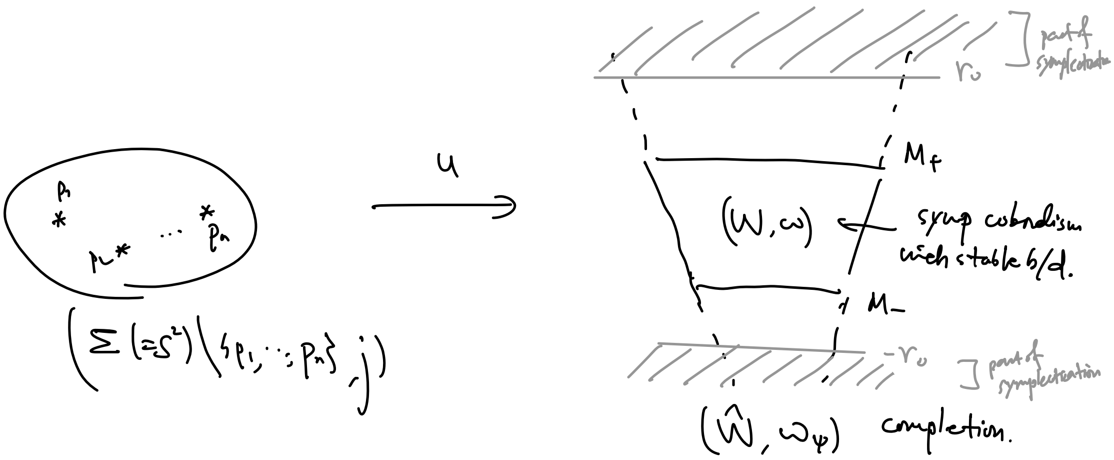
$$= \mathcal{A}_H((x_-, w_-)) - \mathcal{A}_H((x_+, w_+)) + \omega(S) < C$$

$\overset{\text{discussion}}{\underset{\text{above}}{\Rightarrow}}$ finitely many htp class $[\mathrm{im}(u)]$

$\Rightarrow$ finitely many S_i with non-zero a_i.

One can simplify $a_{S_1}T^{S_1} + \cdots$ to be $a_{\lambda_1}T^{\lambda_1} + a_{\lambda_2}T^{\lambda_2} + \cdots$ where $\lambda_n = [\omega](S_n)$. This is the born of Novikov ring in (Ham) Floer theory.

4. Asymptotic behavior near punctures and non-cpt target



Energy constraint: $E_{\psi,\varepsilon,r_0}(u) = \sup_{\varphi \in \mathcal{T}_{\psi,\varepsilon,r_0}} \int_\Sigma u^*\omega_\varphi < +\infty.$

$\leftarrow$ Note that it could be $\int_\Sigma u^*\omega_\psi = +\infty$.

Even near p_i, $E_{\psi,\varepsilon,r_0}(u|_{u,\,\text{NBH of } p_i}) < +\infty$, we need extra argument to get the following conclusion (since the target is non-cpt).

Prop (Lemma 9.14 in [Wen]) $\exists\, C = C(u)$, $|du| \leq C$ in u.

Pf of prop We prove it via three different cases.

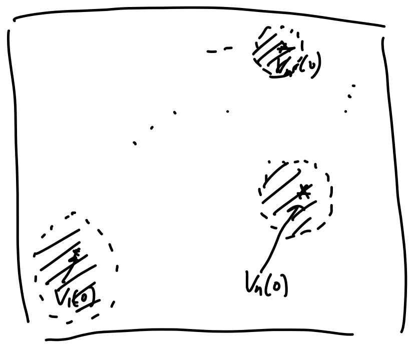
Again, by bubble off analysis: suppose $\underset{=z}{p_i}$ is a bubble pt ($\Leftrightarrow |du(z_n)| \to +\infty$)

for a seq $z_n \to z$, then replacing z_n by $w_n (\to z)$

$$\exists\ v_n : B_{\varepsilon_n |du(w_n)|}(0) \longrightarrow \hat{W} \qquad n \in \mathbb{N}$$

which are J-hol, with $|dv_n| \leq 2$ and $|dv_n(0)| = 1$

Case 1 $\quad \underset{u(w_n)}{\overset{\shortparallel}{v_n(0)}}$ has a bounded subseq.



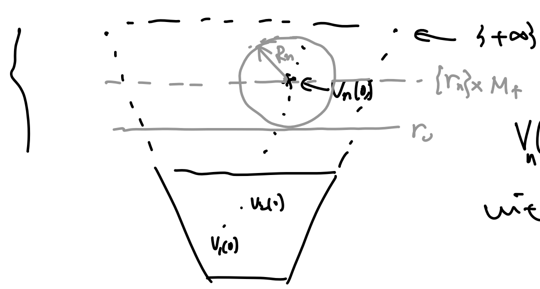
a cpt part in $\hat{W}$

$\overset{\text{compactness}}{\Longrightarrow}$ $v_n \xrightarrow{C^\infty} v_\infty : \mathbb{C} \longrightarrow \hat{W}$ (with $|dv_\infty(0)| = 1$) $\leftarrow$ non-constant

Then finite energy assumption $\Rightarrow$ v_∞ is a constant $\to \Leftarrow$.

Case 2 $\quad v_n(0)$ has a subseq diverging to $\{+\infty\} \times M_+$

$[0,\infty) \times M_+$



$v_n(0) = \{r_n\} \times M_+$

with $r_n \to +\infty$ (as $n \to +\infty$)

Denote R_n the largest radius of a disk $B_{R_n}(0)$ s.t. $\mathrm{im}(v_n|_{B_{R_n}(0)}) \subset [r_0,\infty) \times M_+$

Observe that $R_n \to +\infty$ as $n \to \infty$ (b/c $|dV_n| \leq 2$)

Then in order to control the image of V_n inside a cpt subset, using shifting in $\mathbb{R}$-direction (of symplectization) $\tau(r)$ shifting by r

$$\tilde{V}_n := \tau(-r_n) \circ V_n \big|_{B_{R_n}(0)} : B_{R_n}(0) \longrightarrow \mathbb{R} \times M_+$$

"sliding technique"

Note that $\tilde{V}_n(0)$ is a bounded sequence, so $\exists\, V_\infty : \mathbb{C} \to \mathbb{R} \times M_+$ is J_+-hol where J_+ is the extension of $J|_{[r_0,\infty)\times M_+}$ (by $\mathbb{R}$-invariance).

Now we claim that $\exists\, a \in (-\varepsilon, \varepsilon)$ s.t. $\forall\, \varphi : \mathbb{R} \to (a, \varepsilon)$, $\varphi' > 0$,

$$\int_{\mathbb{C}} V_\infty^* \left(\omega_+ + d(\varphi(r)\lambda_+)\right) < +\infty.$$

(Recall that in $(\hat{W}, \omega_\psi)$, in $[r_0,\infty) \times M_+$, $\omega_\psi = \omega_+ + d(\psi(r)\lambda_+)$.)

proof of claim: Pick any $a > \psi(r_0)$, then for any such φ as above

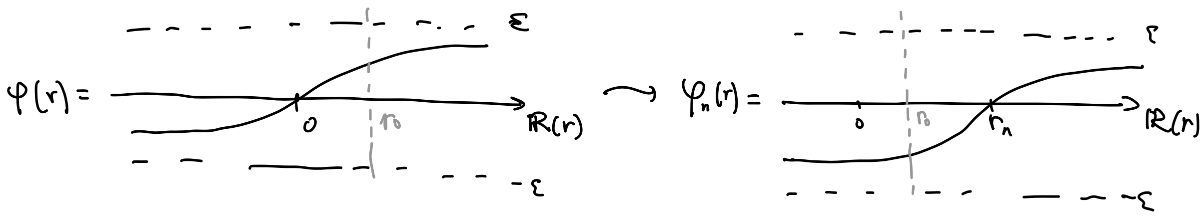
$$\int_{B_R(0)} V_\infty^* \left(\omega_+ + d(\varphi(r)\lambda_+)\right)$$

$$= \lim_{n\to\infty} \int_{B_R(0)} \tilde{V}_n^* \left(\omega_+ + d(\varphi(r)\lambda_+)\right)$$

$$= \lim_{n\to\infty} \int_{B_R(0)} V_n^* \left(\tau(-r_n)^* \left(\omega_+ + d(\varphi(r)\lambda_+)\right)\right)$$

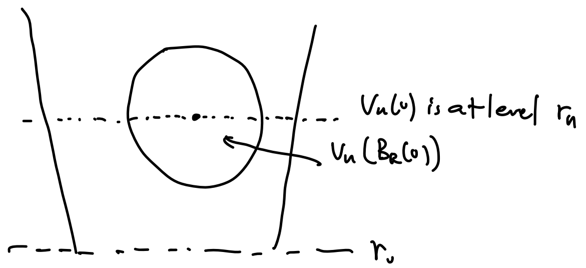
$$= \lim_{n\to\infty} \int_{B_R(0)} V_n^* \left(\omega_+ + d(\varphi_n(r)\lambda_+)\right) \qquad (*)$$

where $\varphi_n(r) = \varphi(r - r_n)$.



For this R, when $n \gg 1$, $B_R(0) \subset B_{R_n}(0)$, so

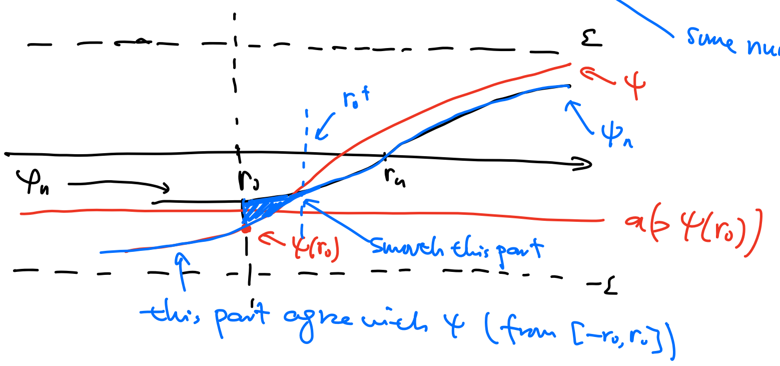
$$V_n(B_R(0)) \subset V_n(B_{R_n}(0)) \subset [r_0, \infty) \times M_+.$$



$\Rightarrow$ for integration in $(*)$ we only need information of $\varphi_n(r)$ for $r \geq r_0$

$\Rightarrow$ for each n, $\exists\, \psi_n \in T_{\psi,\varepsilon,r_0}$ s.t. $\psi_n\big|_{[r_0^+,\infty)} = \varphi_n\big|_{[r_0^+,\infty)}$

same number $r_0^+ > r_0$



Take ψ_n = an extension of $\varphi_n\big|_{[r_0^+,\infty)}$ (s.t. it agrees with ψ in $[-r_0, r_0]$

$\Rightarrow \psi_n \in T_{\psi,\varepsilon,r_0}$)

Rmk One can take smoothing in the "blue" region so r_0^+ can be taken arbitrarily close to r_0.