The uniform upper bound of $E(u_n)$ also implies that $b < +\infty$.

- Since there are only finitely many bubble pt. for each $z^{(i)}$ and its NBH (disjoint from other NBHs of $z^{(j)} \neq z^{(i)}$), carry out the renormalization trick:

$$\exists \text{ J-hol spheres } v^{(i)}: (S^2, j) \longrightarrow (M, \omega, J).$$

In particular, along the process of getting $v^{(i)}$, $\exists$ further bubble pts for seq of J-hol v_n.

- Finally, the top conclusion is derived in a straightforward way. For a precise argument, see Thm 5.2.2 in (ii) in McDuff-Salamon's big book. $\square$

<u>Rmk</u> A less obvious observation from the last top conclusion in Thm 13 that when $n \gg 1$ the homology class $[u_n(\Sigma)]$ is constant (since $H^2(M; \mathbb{Z})$ is a discrete set).

<u>Cor</u>. Let K be a cpt metric space and $\sigma: K \xrightarrow{k \longmapsto \sigma(k)} \mathcal{J}(M, \omega)$ (C^∞-top) be a continuous map. Then for every $C > 0$, $\exists$ only finitely many homology classes $A \in \pi_2(M)$ with

$$\langle [\omega], A \rangle \leq C$$

that can be represented by $J_{\sigma(k)}$-hol spheres for some $k \in K$.

pf. Suppose not, $\exists$ a seq of $K_n \in K$ and $J_{\sigma(K_n)}$-hol curves

$$u_n : (S_2, j) \longrightarrow (M, \omega, J_{\sigma(K_n)}) \quad \text{with} \quad [u_n(S_2)] = A_n \quad \text{and}$$

A_n are all different in $\pi_2(M)$.

$\overset{\text{by Thm}}{\Longrightarrow}$ $\exists$ a subseq (still denoted by) $u_n \longrightarrow u$ and (when $n >> 1$)

$\nwarrow$ J-hol for $J_{\sigma(K_n)} \to J$.

$[u_n(S_2)]$ is stable. $\rightarrow \leftarrow$. $\square$

3. Another type of convergence (or compactness)

Thm (Cor 3.4 in Salomon's Floer homology notes) Fix $x_\pm$ closed Ham orbits of Ham system (M, ω, H, J). If $\{u_n : \mathbb{R}\times S^1 \to M\}_{n\in\mathbb{N}}$ are a seq of Floer cylinder connecting $x_\pm$ with a uniform upper bd of energies, then $\exists$ a seq of finitely many closed Ham orbits $x_0, \dots, x_m$ (with $x_0 = x_-$ and $x_m = x_+$) and finitely many Floer cylinders $\{u^{(i)} : \mathbb{R}\times S^1 \longrightarrow M\}_{i=1}^{m}$ s.t. $u^{(i)}$ connects x_{i-1} and x_i and $\exists$ finitely many seq $s_n^{(i)} \in \mathbb{R}$ for each $i = 1, \dots, m$ s.t. $u_n(s + s_n^{(i)}, t) \xrightarrow{C^\infty_{loc}} u^{(i)}$ except finitely many bubble pts. Moreover

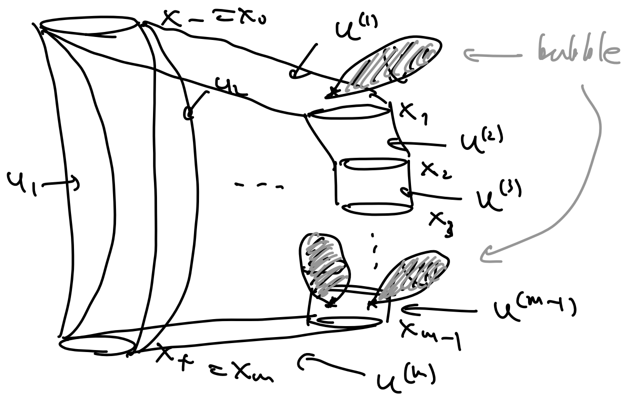
$$\lim_{n\to\infty} [\mathrm{im}(u_n)] = [\mathrm{im}(u_\infty)]$$

$$= [\mathrm{im}(u^{(1)}) \# \cdots \# \mathrm{im}(u^{(m)})] + [\text{bubbles}]$$

$\swarrow$ J-hol spheres

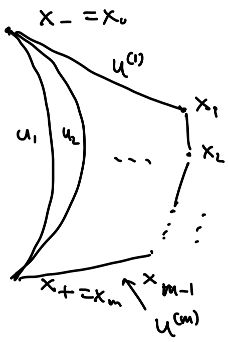
in homotopy class $\pi_2(M; x_\pm)$.

Here is the picture:



Rmk: bubbles shows up naturally in the interior, when the new phenomenon occurs at the asymptotic ends.

Let's demonstrate the proof of this in Morse setting (statement):



- $x_-, x_+ \in \mathrm{Crit}(F)$ for a Morse fcn F on (M, g)
- u_n smooth

- $x_0 = x_-, x_1, \cdots, x_{m-1}, x_m = x_+ \in \mathrm{Crit}(F)$
- $u^{(i)}$ neg. gradient flow line of F connecting x_{i-1} and x_i
- $\exists\, S_n^{(i)} \in \mathbb{R}$ s.t. $u_n(s + S_n^{(i)}) \xrightarrow{C^\infty_{loc}} u^{(i)}$ for each $i = 1, \cdots, m$.

Rmk Since the flow lines in Morse setting are 1-dim'l, there will be no bubbles.

Lemma Given gradient flow lines $\{u_n : \mathbb{R} \to (M, g, F)\}_n$ (cpt), there exists a subseq (still denoted by) u_n and a gradient flow line u st. $u_n \xrightarrow{C^\infty_{loc}} u$, i.e. $\forall R > 0$, $u_n|_{[-R,R]} \xrightarrow{C^\infty} u|_{[-R,R]}$.

$*$ Recall Arzelà-Ascoli Thm in analysis:

$$\underbrace{\text{uniformly bounded}}_{C^0\text{-bound}} + \underbrace{\text{uniform equicontinuous}}_{C^1\text{-bound}} \Longrightarrow \exists \text{ uniform } C^0 \text{ convergence subsequence.}$$

(locally if domain is non-cpt and the limit is cont.)

pf. - Since M is cpt, C^0-bound condition is satisfied.

- Also since M is cpt and Morse fcn F is at least C^2, we know

$\exists C$ s.t. $\|\nabla F(x)\|_g \leq C$ for any $x \in M$.

$\Rightarrow$ For any n, any $s_1, s_2 \in \mathbb{R}$,

$$d(u_n(s_1), u_n(s_2)) \leq \int_{s_1}^{s_2} \|\partial_s u_n(\sigma)\|_g \, d\sigma$$

$$= \int_{s_1}^{s_2} \|\nabla F(u_n(\sigma))\|_g \, d\sigma \leq \underbrace{C(s_1 - s_2)}_{\text{this is ind of } n}$$

(In other words, $\forall \varepsilon > 0$, $\exists \underset{= \frac{\varepsilon}{C}}{\delta}$ s.t. $d(u_n(s_1), u_n(s_2)) < \varepsilon$ whenever $|s_1 - s_2| < \frac{\varepsilon}{C}$.

Here, δ only depends on ε, not on s_1, s_2 and n.)

So $\{u_n\}_n$ is uniform equicontinuous $\Rightarrow$ $\exists$ subseq $u_n \xrightarrow{C^0_{loc}} u$ ← ONLY C^0.

- We will use gradient flow equation inductively.

For the subseq $\{u_n\}$ obtained in the previous step,

$$\dot{u}_n(s) = -\nabla F(u_n(s)) \xrightarrow{C^0_{loc}} \nabla F(u(s))$$

$\Rightarrow$ $u \in C^1(\mathbb{R}, M)$ and $\dot{u}(s) = \nabla F(u(s))$, i.e. u is a gradient flowline.

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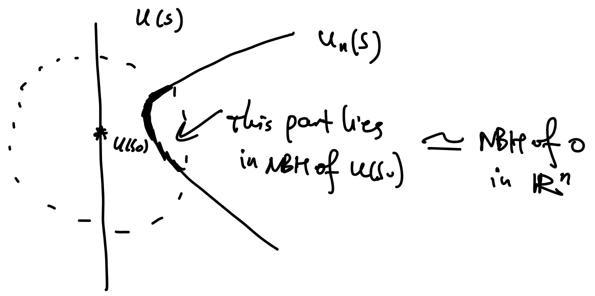
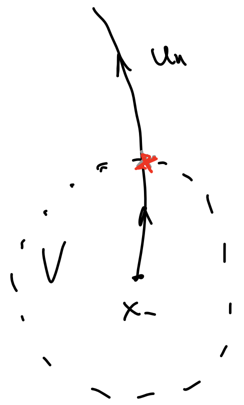


$\Rightarrow$ $\partial_s^k u_n(s) = f(u_n, \partial_s u_n, \dots, \partial_s^{k-1} u_n)$

$\longrightarrow f(u, \partial_s u, \dots, \partial_s^{k-1} u)$

$\Rightarrow$ C^k-converges. □

Now, let's confirm the conclusion of "broken Morse flowline" above.



choose a NBH of x_- so that no other crit(F) lies inside.

$\Rightarrow s_n' = \inf\{ s \in \mathbb{R} \mid u_n(s) \notin V \}$

= the first exit time for flowline u_n for each $n \geq 1$.

(Note that $s_n' < +\infty$).

Then for seq $\{ u_n(s + s_n') : \mathbb{R} \to M \}_n$, it is a seq of gradient flowline of F on (M, g) (since shifting constant does NOT affect the gradient equation).

$\xRightarrow[\text{above}]{\text{Lemma}}$ $\exists$ subseq $u_n(s + s_n') \xrightarrow{C^\infty_{loc}} u^{(1)}$ for some gradient flowline $u^{(1)}$ of F.

$$\text{Moreover,} \quad u^{(1)}(0) \overset{C^0\text{-convergence}}{=} \lim_{n \to \infty} u_n(s_n') \in \partial V$$

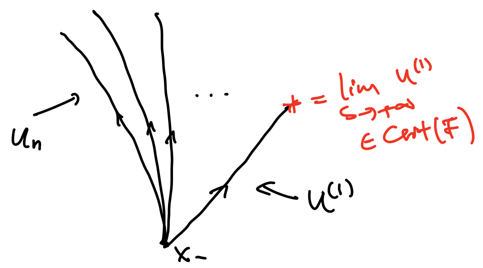
so $u^{(1)}(0) \notin \text{Crit}(F)$ and then $u^{(1)}$ is not constant.

Furthermore, for $\forall s < 0$, $u_n(s + s_n') \in V$ for any n, so

$$u^{(1)}(s) = \lim_{n \to \infty} u_n(s + s_n') \in V$$

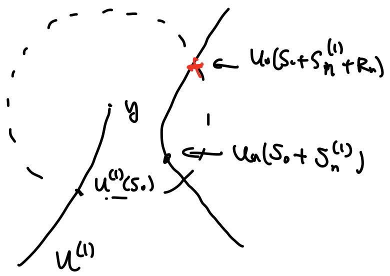
plus, we know (since $u^{(1)}$ is a gradient flowline), $\lim_{s \to -\infty} u^{(1)}(s) =$ some crit pt of F.

$$\Rightarrow \lim_{s \to -\infty} u^{(1)}(s) = x_-$$



Rmk. If $\lim_{s\to+\infty} u^{(1)}(s) = x_+$, then the limit is a "non-broken" gradient flowline (which is OK!).

If $\lim_{s\to+\infty} u^{(1)}(s) = y \ (\neq x_+) \in \mathrm{Crit}(F)$, then consider a sufficiently small NBH W of y: by def $\exists\, s_0 \in \mathbb{R}$ s.t. $\forall s \geq s_0$, $u^{(1)}(s) \in W$.



$\overset{n \gg 1}{\Longrightarrow}$ $u_n(s_0+s_n^{(1)}) \in W$

Similarly as above, define

$$R_n := \inf\{ r \geq 0 \mid u_n(s_0+s_n^{(1)}+r) \notin W \}$$

which is finite b/c $u_n \xrightarrow{s\to+\infty} x_+ \ (\neq y)$.

But $R_n \to +\infty$ as $n \to +\infty$ (due to C^∞-convergence).

Define $s_n^{(2)} := s_0 + s_n^{(1)} + R_n$. Then consider $u_n(s+s_n^{(2)})$.

$\xRightarrow[\text{again}]{\text{Lemma}}$ $\exists$ $u_n(s+s_n^{(2)}) \xrightarrow{C^\infty_{loc}} u^{(2)}(s)$ for another non-constant gradient flow of F.

Moreover, $\lim_{s\to-\infty} u^{(2)}(s) = \lim_{s\to-\infty} u_n(s+s_n^{(2)}) = y$ ← check DIY.

Finally, we observe that this processes will terminate since there are only finitely many critical pts (and F needs to increase along any gradient flowline). $\square$