- $\forall$ cpt $K \subset \mathbb{C}$, $\exists\, n \gg 1$ s.t. $B_{\varepsilon_n |du_n(w_n)|}(0) \supset K$.

- $\forall n$, $dv_n(0) = \dfrac{1}{|du_n(w_n)|} du_n(w_n) \Longrightarrow |dv_n(0)| = 1$

- $\forall n$, $\forall w \in B_{\varepsilon_n |du_n(w_n)|}(0)$, we have

$$|dv_n(w)| = \frac{1}{|du_n(w_n)|}\left| du_n\Big(\overbrace{w_n + \frac{w}{|du_n(w_n)|}}^{\in B_{\varepsilon_n}(w_n)}\Big)\right| \underset{\text{by (iv)}}{\le} \frac{2|du_n(w_n)|}{|du_n(w_n)|} = 2. \quad (*)$$

Also, $$E(v_n) = \int_{B_{\varepsilon_n |du_n(w_n)|}(0)} |dv_n|^2 \, dvol = \int_{B_{\varepsilon_n |du_n(w_n)|}(0)} |du_n(-)|^2 \frac{dvol}{|du_n(w_n)|^2}$$

$$\overset{\text{change of variable}}{=} \int_{B_{\varepsilon_n}(w_n)} |du_n|^2 \, dvol < \underset{\text{uniform upper bd}}{C}.$$

and the third item right above implies that $|dv_n|_{L^\infty}$ is uniformly bd.

- We claim (Exe $\sim$ FACT) $\exists$ subseq of $v_n$ that converges to a J-hol map $v_\infty : (\mathbb{C}, j_{std}) \xrightarrow{C^\infty_{loc}} (M, J)$ ( $C^\infty_{loc}$ meaning smoothly on any cpt subset of $\mathbb{C}$).

(we will come back to this when we discuss "compactness" later).

Then $E(v_\infty) = \lim_{n\to+\infty} E(v_n) < C$ and $|dv_\infty(0)| = \lim_{n\to\infty} |dv_n(0)| = 1$

($\Rightarrow$ $v_\infty$ is not constant)

Consider $\tilde{v}_\infty : \mathbb{C} \setminus \{0\} \to M$ by $\tilde{v}_\infty(z) = v_\infty(1/z)$

$\Rightarrow E(\tilde{v}_\infty) < C$ and then $\tilde{v}_\infty$ extends to a J-hol map $\mathbb{C} \to M$.

by Removal of Singularities.

$\underset{v_\infty \text{ and } \tilde{v}_\infty}{\overset{\text{glue}}{\Longrightarrow}}$ $v : (\underset{\mathbb{C}\cup\{\infty\}}{S^2}, j_{std}) \to (M, J)$ J-hol

$\Longrightarrow \quad E(v) \geq \hbar \quad$ (since $v$ is nonconstant b/c $v|_{\mathbb{C}} = v_\infty : \mathbb{C} \to M$)

$\Longrightarrow \quad E\left(u_n\big|_{B_{\varepsilon_n}(w_n)}\right) = E(v_n) \geq \hbar - \varepsilon.$

For any given NBH $\mathcal{U}$ of $z$, $\exists\, n$ s.t. $B_{\varepsilon_n}(w_n) \subset \mathcal{U}$, so

$$\varliminf_{n\to\infty} E\left(u_n\big|_{B_{\varepsilon_n}(w_n)}\right) \geq \hbar \quad \Longrightarrow \quad z \text{ is a bubble pt.} \qquad \square$$

<u>Rmk</u>* The uniform bound on energy $E(u_n)$ is to guarantee the application of the removal of singularities.

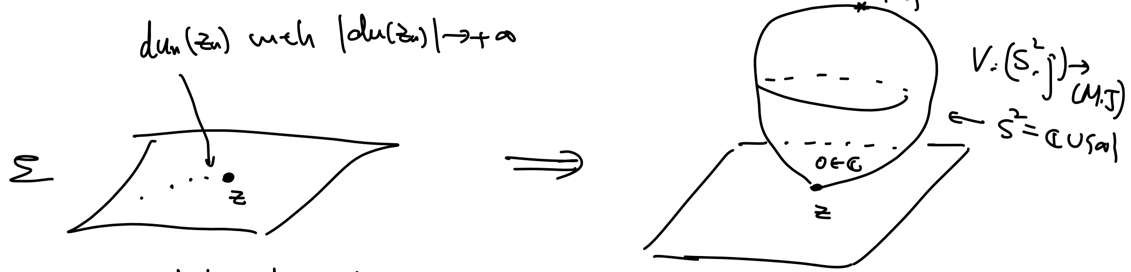
<u>Rmk</u> The compactness in Claim above comes from two conditions:
(cf. McDuff-Salamon big book Thm B.4.2)

target $M$ is compact $+$ $\underbrace{\text{uniform upper bound of } \|dv_n\|}_{C^1\text{-bound}}$

This can be weakened that $\mathrm{im}(u_n)$ or $\mathrm{im}(v_n) \subset$ a cpt subset in $M$.

<u>Rmk</u> The same argument works for $J_n$-hol curve $u_n$ with $J_n \to J$ (and still bounded energy $E(u_n)$) $\Longrightarrow \exists\, v : (S^2, j) \to (M, J)$ which is $J$-hol.

<u>Rmk</u> Here is a schematic picture for the "bubble" $v$:




computational picture

domain of the limit curve is $\Sigma \underset{\substack{z \\ \text{bubble pt}}}{\cup} S^2$

Rmk By discussion above, $z \in \Sigma$ is a bubble pt of a single $u: (\Sigma, j) \to (M, J)$ iff $\exists\, z_n \to z$ in $\Sigma$ s.t.

$$|du(z_n)| \longrightarrow +\infty.$$

We claim that for closed $\Sigma$ (and cpt $M$), $u$ does not have any bubble pt: $[u(\Sigma)] \in H^2(M; \mathbb{Z})$ and recall

$$E(u) = \int_\Sigma u^*\omega = \langle [\omega], [u(\Sigma)] \rangle < \infty$$

$\Rightarrow$ If $z$ is a bubble pt, then any NBH $U$ of $z$ has $E(u|_U) < \infty$.

Then $|du(z_n)| \to +\infty$ implies that

$$\underbrace{\sum_{n_i}}_{\infty\text{-ly many sum!}} \int_{B_{\varepsilon_{n_i}|du(w_{n_i})|}(0)} v_{n_i}^*\omega = \sum_{n_i} \int_{B_{\varepsilon_{n_i}}(w_{n_i})} u^*\omega \leq \int_U u^*\omega < +\infty$$

$\varepsilon_{n_i}|du(w_{n_i})| \to +\infty$

$w_{n_i} \to z$ and $\varepsilon_{n_i} \to 0$
so take $\varepsilon_{n_i} << 1$ s.t. $B_{\varepsilon_i}(w_{n_i})$ are disjoint

$$\Rightarrow \int_{B_{\varepsilon_{n_i}|du(w_{n_i})|}(0)} v_{n_i}^*\omega \to 0 \quad \text{as } n_i \to +\infty$$

$$\Rightarrow \int_{\mathbb{C}} v_\infty^*\omega = 0$$ (which contradicts $v_\infty$ is non-constant) //

In other words, if $\int_U u^*\omega < +\infty$ ($U$ = NBH of $z$), then near $z$, $|du| \leq C$ for some $C = C(u)$.

See the end of section 9.1.1 in [Wen]

again, under the hypothesis $M$ is cpt or the image $u$ lies in a cpt subset of $M$.

(cf. $u: (U \setminus \{z\}, j) \to (M, J)$, finite energy $\Rightarrow$ $z$ is removeable.)

## 2. Bubble tree

The following result is a direct cor of the Prop above, which describe the global behavior of the limit of a seq of J-hol curves.

<u>Thm</u> $\{u_n : (\overset{\swarrow \text{closed}}{\Sigma}, j) \longrightarrow (\overset{\swarrow \text{cpt}}{M}, \omega, J_n)\}_n$ $J_n$-hol where $J_n \to J$.

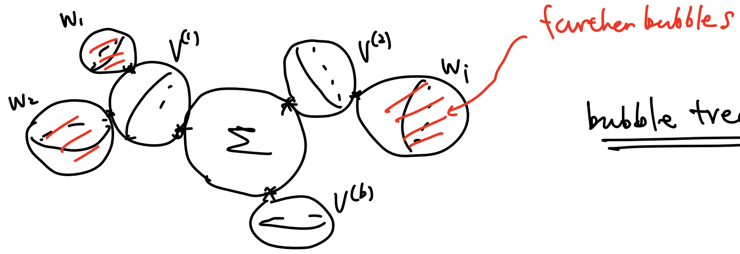
Assume $E(u_n) < C$ for a uniform constant $C$. Then $\exists$ a finite collection of pts $z^{(1)}, \cdots, z^{(b)} \in \Sigma$ and a J-hol curve $u : \Sigma \longrightarrow M$ s.t.

- For any cpt subset $K \subset \Sigma \setminus \{z^{(1)}, \cdots, z^{(b)}\}$, we have $u_n|_K \xrightarrow{C^\infty} u|_K$.
- There are non-constant J-hol sphere $V^{(1)}, \cdots, V^{(b)} : S^2 \to M$ s.t. $V^{(i)}(\infty) = u(z^i)$
- For some finite (possibly empty) collection of non-constant J-hol sphere $w_i : S^2 \to M$ s.t. for each $i$, $w_i(\infty)$ is in the image of either of some other $w_j$ or of some $V^{(i)}$, we have

further steps of bubbles.

$$\lim_{n\to\infty} (u_n)_*([\Sigma]) = u_*([\Sigma]) + \sum_{j=1}^{b} V_*^{(j)}([S^2]) + \sum_i (w_i)_*([S^2])$$ //

If one views $\{u, \{V^{(j)}\}_{i=1}^{b}, \{w_j\}_i\}$ together over a single domain, then this domain looks like




<u>bubble tree</u>

Rmk One can use this bubble tree to define a topology (called Gromov topology). In particular, one defines compactness via sequential compactness + a limit described by J-hol maps over bubble tree.

$\Longrightarrow$ compactification of the moduli space (of J-hol curve)

$\Longrightarrow$ Gromov-Witten invariant ...

For more details, see Chapter 5 in McDuff - Salamon's big book.

Prof of Thm

- Suppose $\sup_n \|du_n\|_{L^\infty(\Sigma)} < +\infty$, then we are DONE (no bubble pt and $b=0$)

  Suppose not, i.e. $|du_n(z_n)| \to +\infty$ for some seq $z_n \to z^{(1)}$ (b/c $\Sigma$ is cpt).

$\overset{\text{Prop}}{\Longrightarrow}$ $z^{(1)} \in \Sigma$ is a bubble pt of $\{u_n\}$.

$\overset{\text{by def}}{\Longrightarrow}$
$$\lim_{n \to} E(u_n|_{\underset{\text{NBH of } z^{(1)}}{U}}) = \lim_{n \to} E(u_n|_{B_\delta(z^{(1)})}) \geq \hbar \ (>0) \qquad (*)$$
for any (small) $\delta > 0$

Then consider $\Sigma \setminus \{z^{(1)}\}$.

- Suppose $\sup_n \|du_n\|_{L^\infty(K)} < C(K)$ uniform constant for every cpt subset $K \subset \Sigma \setminus \{z^{(1)}\}$, then Exe/FACT above implies:

  $\exists$ J-hol $u: \Sigma \setminus \{z^{(1)}\} \to (M, \omega, J)$ and $u_n \xrightarrow{C^\infty_{loc}} u$.

  Then $(*) \Rightarrow \forall n, \ E(u_n|_{K \subset \Sigma \setminus \{z^{(1)}\}}) < C - \hbar$ ← independent of K.

  any cpt K

  $$\Rightarrow E(u|_{K \subset \Sigma \setminus \{z^{(1)}\}}) < C - \hbar$$

$\underset{\text{removal of singularities}}{\Longrightarrow}$ $u$ extends to a J-hol $(\Sigma, j) \to (M, \omega, J)$

- Suppose $\exists$ cpt $K \subset \Sigma \setminus \{z^{(1)}\}$ s.t. $\exists\, z_n \to z^{(2)}$ in $K$ with $|du_n(z_n)| \to +\infty$. In particular, $z^{(2)} \neq z^{(1)}$.

$\Longrightarrow$ $z^{(2)}$ is a bubble pt of $\{u_n\}$

$$\Longrightarrow \quad \varliminf_{n} E\big(u_n|_{\underset{\text{NBH of } z^{(2)}}{U}}\big) = \varliminf_{n} E\big(u_n|_{B_\delta(z^{(2)})}\big) \geq \hbar \ (>0).$$

Then consider $\Sigma \setminus \{z^{(1)}, z^{(2)}\}$.

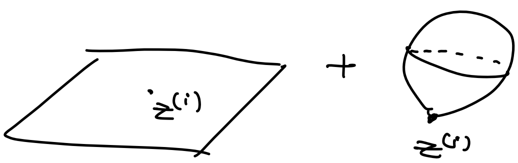
Inductively, we obtain $\Sigma \setminus \{z^{(1)}, \dots, z^{(b)}\}$ and $\underbrace{u : (\Sigma, j) \to (M, \omega, J)}_{\text{after removal of singularities}}$

at this moment we don't know if $b$ is finite or not.

- For each $z^{(i)} \in \Sigma$, consider

$$\lim_{\delta \to 0} \Big( \underbrace{\varliminf_{n \to \infty} E\big(u_n|_{B_\delta(z^{(i)})}\big)}_{\substack{\text{one can take subseq of } \{u_n\} \\ \text{so that } \varliminf_n \text{ above is a limit}}} \Big) =: m(z^{(i)}) \ (\geq \hbar)$$

$$\Longrightarrow \quad \lim_{n \to \infty} E(u_n) = E(u) + \sum_{i=1}^{b} m(z^{(i)})$$




$\{m(z^{(i)})\}_i$ measures the energy lost when removing the singularities.

The uniform upper bound of $E(u_n)$ also implies that $b < +\infty$.

- Since there are only finitely many bubble pt, for each $z^{(i)}$ and its NBH (disjoint from other NBHs of $z^{(j)} \neq z^{(i)}$), carry out the renormalization trick:

$$\exists \text{ J-hol spheres } v^{(i)}: (S^2, j) \longrightarrow (M, \omega, J).$$

In particular, along the process of getting $v^{(i)}$, $\exists$ further bubble pts for seq of J-hol $v_n$.

- Finally, the top conclusion is derived in a straightforward way. For a precise argument, see Thm 5.2.2 in (ii) in McDuff - Salamon's big book. $\square$

<u>Rmk</u> A less obvious observation from the last top conclusion in Thm 13 that when $n \gg 1$ the homology class $[u_n(\Sigma)]$ is constant (since $H^2(M; \mathbb{Z})$ is a discrete set).

<u>Cor</u>. Let $K$ be a cpt metric space and $\sigma: K \xrightarrow{k \longrightarrow \sigma(k)} \mathcal{J}(M, \omega)$ ($C^\infty$-top) be a continuous map. Then for every $C > 0$, $\exists$ only finitely many htp classes $A \in \pi_2(M)$ with

$$\langle [\omega], A \rangle \leq C$$

that can be represented by $J_{\sigma(k)}$-hol spheres for some $k \in K$.