1. Bubbling off analysis

This is a technique introduced by Gromov, and trace back to Sacks-Uhlenbeck (on harmonic maps)

in this section, assume M is cpt.

Def Given a seq of J-hol map $\{u_n : (\Sigma, j) \to (M, \omega, J)\}_{n \in \mathbb{N}}$, a pt $p \in \Sigma$ is called a bubble pt of $\{u_n\}_{n\in\mathbb{N}}$ if for every open NBH U of p in Σ, we have

$$\lim_{\overrightarrow{n}} E(u_n|_U) \; (= \text{Area}(u_n|_U)) \geq \hbar$$

(for a uniform constant $\hbar > 0$)

Here, recall that constant $\hbar$ is the "quantum jump constant" for a non-constant J-hol sphere in (M, ω, J).

Rmk. Def above applies to $\{u_n\} = u$, a single J-hol map.

Rmk If $\exists$ U_0 NBH of $p \in \Sigma$ st. $\|du_n\|_{L^\infty} < C$, then

$$E(u_n|_U) = \int_U |du(z)|^2 \underbrace{dvol}_{\text{fixed}} \leq C^2 \, Vol(U)$$

for any $U \subset U_0$. Then p is not a bubble pt (of $\{u_n\}_{n\in\mathbb{N}}$).

be careful when we use this condition in later proof!

Prop Given J-hol seq $\{u_n\}_n$ with $E(u_n) < C$ uniformly for u_n (for some $C > 0$), Suppose $\exists$ $z_n (\in \Sigma) \to z$ and $|du_n(z_n)| \to +\infty$, then z is a bubble pt of $\{u_n\}_{n\in\mathbb{N}}$.

One of the tricky steps is the following lemma due to Hofer.

Lemma (Lemma 9.4 in [Wen]) (X,d) complete metric space. $\delta>0$, $x\in X$. $f: X\to[0,\infty)$ a continuous fcn. Then $\exists\ \xi\in X$, $\varepsilon>0$ with the following properties

(i) $\varepsilon\leq\delta$

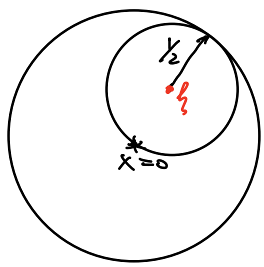
(ii) $d(x,\xi)<2\delta$

(iii) $\varepsilon f(\xi)\geq\delta f(x)$

(iv) $2f(\xi)\geq\sup\limits_{B_\varepsilon(\xi)} f$

Ex $X=\mathbb{R}^2$ $x=0$, $\delta=$ radius 1. $f:\mathbb{R}^2\to[0,\infty)$ is the distance to the origin.

$\delta=1$



(i) $\varepsilon=\frac{1}{2}\leq 1=\delta$

(ii) $d(x,\xi)=d(0,\xi)<2\delta$

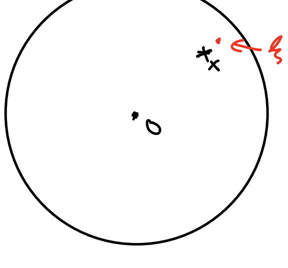
(iii) $\frac{1}{2}f(\xi)=\frac{1}{2}d(0,\xi)=\frac{1}{4}\geq 1.d(0,0)=0.$

(iv) $2f(\xi)=2.d(0,\xi)=1\geq\underbrace{\sup\limits_{B_{\frac{1}{2}}(\xi)} f}_{\text{this reaches the limit.}}$

Note that in this case when ξ is chosen in X (and $\xi\neq x=0$), $\exists\ \varepsilon$ small enough s.t. (i) $\to$ (iv) hold (in particular, (iv) holds).

In the same setting, but with x close to $\partial B_1(0)$:

$\delta=1$



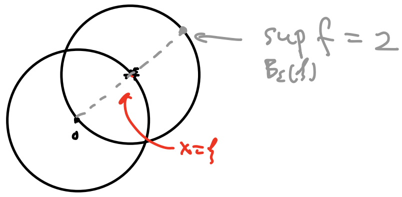
(i) $\varepsilon\leq 1=\delta$

(ii) $d(x,\xi)<1<2\delta$

(iii) $\varepsilon f(\xi)\geq 1.d(x,0)$ ← ε should be large enough (close to 1)

(iv) $\underset{\underset{2}{\text{ss}}}{2f(\xi)}\geq\sup\limits_{B_\varepsilon(\xi)} f$

For condition (iv), the extreme picture is



<u>Rmk</u> This also shows that "2" in (iv) is nec.

<u>Pf of Lemma</u> Suppose any pt $p \in B_\delta(x)$ satisfies $2f(x) \geq f(p)$, so

$$2f(x) \geq \sup_{B_\delta(x)} f.$$

then one can take $\xi = x$ and $\varepsilon = \delta$, then (i) – (iv) satisfy. ✓

Suppose not, then $\exists\, p_1 \in B_\delta(x)$ s.t. $2f(x) < f(p_1)$. Set $\varepsilon_1 = \frac{\delta}{2}$

Then repeat the argument above for $B_{\varepsilon_1}(p_1)$:

- $\forall p \in B_{\varepsilon_1}(p_1)$ satisfies $2f(p_1) \geq f(p)$ $\left(\Rightarrow 2f(p_1) \geq \sup_{B_{\varepsilon_1}(p_1)} f\right)$

  then take $\xi = p_1$ and $\varepsilon = \varepsilon_1$ then

  (i) $\varepsilon = \varepsilon_1 = \frac{\delta}{2} < \delta$.

  (ii) $d(x, \xi) = d(x, p_1) \leq \delta < 2\delta$

  (iii) $\varepsilon f(\xi) = \frac{\delta}{2} f(p_1) \geq \delta f(x)$ ✓

  (iv) $2f(\xi) = 2f(p_1) \geq \sup_{B_{\varepsilon_1}(p_1)} f(p)$ ✓ ⟵ by our assumption

- If not, then $\exists\, p_2 \in B_{\varepsilon_1}(p_1)$ s.t. $2f(p_1) < f(p_2)$. $\Rightarrow$ set $\varepsilon_2 = \frac{\varepsilon_1}{2} = \frac{\delta}{2^2}$

  and repeat the argument above for $B_{\varepsilon_2}(p_2)$

  ...

$\Rightarrow$ either we finish the proof at some step, or

$$\exists \text{ seq } \underbrace{p_{n+1} \in B_{\varepsilon_n}(p_n) = B_{\frac{\delta}{2^n}}(p_n)}_{\Leftrightarrow\ d(p_n, p_{n+1}) \leq \frac{\delta}{2^n}} \text{ and } \underbrace{2f(p_n) < f(p_{n+1})}_{\Rightarrow\ f(p_n) \text{ diverge.}}$$

This is impossible since $f(p_*) = \infty$ when $p_* = \lim p_n \in X$

(as $\{p_n\}$ is a Cauchy sequence). $\square$

Back to the proof of prop.

<u>Pf</u> • We will set up the following correspondence (applying lemma to each $u_n(z_n)$ inductively):

$$\left.\begin{array}{l} (X, d) \ \text{NBH (chart) of } z \\ \delta = \dfrac{1}{|du_n(z_n)|^{1/2}} \\ x = z_n \\ f: X \to [0,\infty) \text{ by } |du_n(w)| \text{ for } w \in X \end{array}\right\} \Rightarrow \begin{array}{l} \exists \xi = w_n, \ \varepsilon = \varepsilon_n \ \text{s.t.} \\ \text{(i) } \varepsilon_n \leq \dfrac{1}{|du_n(z_n)|^{1/2}} \\ \text{(ii) } d(z_n, w_n) < \dfrac{2}{|du_n(z_n)|^{1/2}} \\ \text{(iii) } \varepsilon_n |du_n(w_n)| \geq \dfrac{|du_n(z_n)|}{|du_n(z_n)|^{1/2}} = |du_n(z_n)|^{1/2} \\ \text{(iv) } 2|du_n(w_n)| \geq \sup\limits_{B_{\varepsilon_n}(w_n)} |du_n(w)| \end{array}$$

In other words, we replace $\{z_n\}$ with $\{w_n\}$ s.t.

(ii) $\Rightarrow$ $w_n \longrightarrow z$ by triangle inequality and $|du_n(z_n)| \to +\infty$.

(i), (iv) $\Rightarrow$ up to factor 2, w_n obtain the local maxima within $B_{\varepsilon_n}(w_n)$.

• Define
$$V_n : B_{\varepsilon_n |du_n(w_n)|}(0) \longrightarrow M$$
$$w \longrightarrow u_n\Big(w_n + \underbrace{\frac{w}{|du_n(w_n)|}}\Big)$$

By (iii), the radius $\varepsilon_n |du_n(w_n)| \to +\infty$. $\qquad \left|\dfrac{w}{|du_n(w_n)|}\right| \leq \varepsilon_n \to 0$

Here are some basic observations:

- $\forall$ cpt $K \subset \mathbb{C}$, $\exists\, n \gg 1$ s.t. $B_{\varepsilon_n |du_n(w_n)|}(0) \supset K$.

- $\forall n$, $dv_n(0) = \frac{1}{|du_n(w_n)|} du_n(w_n) \Rightarrow |dv_n(0)| = 1$

- $\forall n$, $\forall w \in B_{\varepsilon_n |du_n(w_n)|}(0)$, we have

$$|dv_n(w)| = \frac{1}{|du_n(w_n)|} \left| du_n\Big(\overbrace{w_n + \frac{w}{|du_n(w_n)|}}^{\in B_{\varepsilon_n}(w_n)}\Big) \right| \overset{\text{by (iv)}}{\le} \frac{2\,|du_n(w_n)|}{|du_n(w_n)|} = 2. \quad (*)$$

Also, $E(v_n) = \int_{B_{\varepsilon_n |du_n(w_n)|}(0)} |dv_n|^2 \, d\mathrm{vol} = \int_{B_{\varepsilon_n |du_n(w_n)|}(0)} |du_n(-)|^2 \frac{d\mathrm{vol}}{|du_n(w_n)|^2}$

$$\underset{\text{of variable}}{\overset{\text{change}}{=}} \int_{B_{\varepsilon_n}(w_n)} |du_n|^2 \, d\mathrm{vol} < \underset{\text{uniform upper b/d}}{C}.$$

and the third item right above implies that $|dv_n|_{L^\infty}$ is uniformly b/d.

- We claim $\overset{\sim\text{FACT}}{(\text{Exe})}$ $\exists$ subseq of v_n that converges to a J-hol map $v_\infty : (\mathbb{C}, j_{std}) \xrightarrow{C^\infty_{loc}} (M, J)$ (C^∞_{loc} meaning smoothly on any cpt subset of $\mathbb{C}$).

(we will come back to this when we discuss "compactness" later).

Then $E(v_\infty) = \lim_{n \to +\infty} E(v_n) < C$ and $|dv_\infty(0)| = \lim_{n \to \infty} |dv_n(0)| = 1$

($\Rightarrow$ v_∞ is not constant)

Consider $\tilde{v}_\infty : \mathbb{C} \setminus \{0\} \to M$ by $\tilde{v}_\infty(z) = v_\infty(1/z)$

$\Rightarrow E(\tilde{v}_\infty) < C$ and then $\tilde{v}_\infty$ extends to a J-hol map $\mathbb{C} \to M$.

by Removal of Singularities.

$\underset{v_\infty \text{ and } \tilde{v}_\infty}{\overset{\text{glue}}{\Longrightarrow}}$ $v : (\underset{\mathbb{C} \cup \{\infty\}}{S^2}, j_{std}) \to (M, J)$ J-hol