$$W(s_0,t_0) = V(0,0) \leq \frac{1}{\pi r^2}\int_{B(0,r)} V$$

$$= \frac{1}{\pi r^2}\int_{B((s_0,t_0),r)} W + \frac{b}{4\pi r^2}\int_{B(0,r)} (s^2+t^2)\,ds\,dt$$

$$= \frac{b}{4\pi r^2}\int_0^r\int_0^{2\pi} \rho^2\,\rho\,d\rho\,d\theta + \frac{1}{\pi r^2}\int_{B((s_0,t_0),r)} W$$

$$= \frac{b}{\cancel{4\pi r^2}}\, 2\cancel{\pi}\cdot\frac{1}{4}\overset{r^2}{\cancel{r^4}} + \frac{1}{\pi r^2}\int_{B((s_0,t_0),r)} W$$

$$= \frac{br^2}{8} + \frac{1}{\pi r^2}\int_{B((s_0,t_0),r)} W. \qquad \square$$

Based on these lemmas, we have the following "ptwise" estimation:

<u>Prop</u> Suppose $\xi : \mathbb{R}\times S^1 \to \mathbb{R}^{2n}$ satisfies $D\xi = 0$, then $\forall (s_0,t_0)\in \mathbb{R}\times S^1$, we have

$$\|\xi(s_0,t_0)\|^2 \leq \frac{8a}{\pi}\int_{B((s_0,t_0),1)} \|\xi(s,t)\|^2\,ds\,dt$$

for some $a \geq 1$.

$\leftarrow$ Here we can take the radius 1 smaller (to fit into $\mathbb{R}\times S^1$ if needed) then the constant $\frac{8a}{\pi}$ will be changed or rescaled accordingly.

Note that this implies the desired exponential decay of $\|\xi(s_0,t_0)\|$:

(s_0, t) $t\in\mathbb{R}/\mathbb{Z}$

$$\Longrightarrow \int_{s_0-\varepsilon}^{s_0+\varepsilon}\int_0^1 \|\xi(s,t)\|^2\,dt\,ds$$

$$\leq c\int_{s_0-\varepsilon}^{s_0+\varepsilon} e^{-\delta s}\,ds$$

$$= \frac{c}{-\delta}\left(e^{-\delta(s_0+\varepsilon)} - e^{-\delta(s_0-\varepsilon)}\right)$$

$$= c' e^{-\delta s_0}\cdot O(\varepsilon).$$

apply $\int_0^1 \|\xi(s,t)\|^2\,dt \leq c e^{-\delta|s|}$ (uniform in s) to a NBH of s_0

Within the red shaded region, for any (s_0,t_0), $\exists$ NBH $B((s_0,t_0),r)$ $\subset$ red shaded region, so by prop above,

$$\|\zeta(s_0,t_0)\|^2 \le C'' \int_{B((s_0,t_0),r)} \|\zeta(s,t)\|^2 \, ds\, dt$$

depending on ε.

$$\le C''' e^{-\delta s_0}$$

where $C''' = C'''(\delta, \varepsilon)$, independent of $s_0 \in \mathbb{R}$. Take ε as a uniform width, we get $\|\zeta(s_0,t_0)\| \le \sqrt{C'''}\, e^{-\frac{\delta}{2}|s_0|}$ for $s_0 \in \mathbb{R}$. DONE.

<u>Pf of prop above</u>: The only trouble is that the conclusion of <u>Lemma 1</u> do NOT fit into the hypothesis of <u>Lemma 2</u>. Here is a trick: denote $w(s,t) = \|\zeta(s,t)\|^2$

Consider $f : [0,1] \to \mathbb{R}$ defined by

this decreases when $\rho \nearrow$ — This increases when $\rho \nearrow$

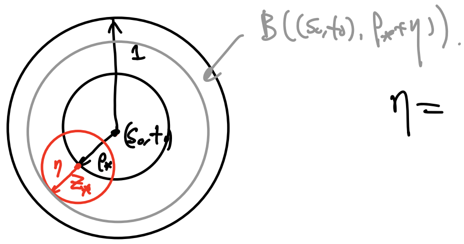
$$\rho \longmapsto (1-\rho)^2 \sup_{B((s_0,t_0),\rho)} w$$

Then f is continuous with $f(0) = \underbrace{w(s_0,t_0)}_{>0}$ and $f(1) = 0$.

$$\Longrightarrow \exists\ \rho_* \in [0,1] \text{ s.t. } \max_{\rho \in [0,1]} f = f(\rho_*)$$

$$= (1-\rho_*)^2 \sup_{B((s_0,t_0),\rho_*)} w = (1-\rho_*)^2 w(z_*)$$

for some $z_* \in B((s_0,t_0),\rho_*)$. Now choose $\eta \in (0,1)$ so that we have the following picture:



$$\eta = \frac{1-\rho_*}{2} \quad \text{e.g.}$$

$$\sup_{B(z_*,\eta)} w \le \sup_{B((s_0,t_0),\,\rho_*+\eta)} w = \frac{f(\rho_*+\eta)}{(1-\rho_*-\eta)^2} = \frac{4f(\rho_*+\eta)}{(1-\rho_*)^2}$$

$$\le \frac{4f(\rho_*)}{(1-\rho_*)^2} \le 4w(z_*)$$

By Lemma 1 (applied to $B(z_*,\eta)$), we get

$$\Delta w \ge -aw \ge \underbrace{-a\cdot 4w(z_*)}_{=-b}$$

for $B(z_*,r)$, $r\le\eta$

Then Lemma 2 applies and we get

$$w(z_*) \le \frac{a\cdot 4w(z_*)\,r^2}{8} + \frac{1}{\pi r^2}\int_{B(z_*,r)} w$$

b/c w is non-negative $\rightarrow$

$$\le \frac{a\,w(z_*)\,r^2}{2} + \frac{1}{\pi r^2}\int_{B((s_0,t_0),1)} w$$

Now, choose $r = \eta/\sqrt{a}$, then

$$w(z_*) \le \frac{w(z_*)\,\eta^2}{2} + \frac{a}{\pi\eta^2}\int_{B((s_0,t_0),1)} w$$

$$\le \frac{w(z_*)}{2} + \frac{a}{\pi\eta^2}\int_{B((s_0,t_0),1)} w$$

$$\Longrightarrow \quad w(z_*) \le \frac{2a}{\pi\eta^2}\int_{B((s_0,t_0),1)} w$$

$$\Longleftrightarrow \quad \eta^2 w(z_*) \le \frac{2a}{\pi}\int_{B((s_0,t_0),1)} w$$

Then $$w(s_0,t_0) = f(0) \le f(\rho_*) = (1-\rho_*)^2 w(z_*)$$

$$= 4\eta^2 w(z_*) \le \frac{8a}{\pi}\int_{B((s_0,t_0),1)} w. \quad \square$$