Under a certain global trivialization of $\begin{matrix} u^*TM \\ \downarrow \\ \mathbb{R}\times S^1 \end{matrix}$, D_u is simplified as

$$D(\xi) = \partial_s \xi + J_0 \partial_t \xi + S\cdot\xi \qquad (\circledast)$$

$\xi : \mathbb{R}\times S^1 \to \mathbb{R}^{2n}$

where $S(s,t) : \mathbb{R}\times S^1 \longrightarrow M_{2n\times 2n}(\mathbb{R})$ satisfies $\lim_{s\to\pm\infty} S(s,t) = S_\pm(t)$, $S_\pm(t)$ are symmetric matrices. Moreover, $\lim_{s\to\pm\infty} \frac{\partial S}{\partial s}(s,t) = 0$ (uniform in t).

The proof of ② $\Rightarrow$ ③ lies in the following proposition.

<u>Prop</u> Suppose $\bar\partial_{H,J} u = 0$, and $\xi : \mathbb{R}\times S^1 \to \mathbb{R}^{2n}$ satisfies $D\xi = 0$ and

$$\int_0^1 \|\xi(s,t)\|^2_{g_{J_{std}}} dt \not\to +\infty$$

$g_{J_{std}}$ = Euclidean metric

← this holds automatically by the second condition in item ② in Thm above.

then $\exists\, \delta > 0$ (ind of u) and $C > 0$ s.t. $\underline{\int_0^1 \|\xi(s,t)\|^2 dt} \leq C e^{-\delta|s|}$ $\forall s \in \mathbb{R}$.

<u>Pf</u> Consider

$$f(s) = \frac{1}{2}\int_0^1 \|\xi(s,t)\|^2 dt : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$$

Then

$$f'(s) = \int_0^1 \langle \xi(s,t), \frac{\partial \xi}{\partial s}(s,t)\rangle dt$$

$$f''(s) = \int_0^1 \|\frac{\partial \xi}{\partial s}(s,t)\|^2 dt + \int_0^1 \langle \xi(s,t), \frac{\partial^2 \xi}{\partial s^2}(s,t)\rangle dt$$

Due to $(\circledast)$, $\frac{\partial \xi}{\partial s} = -J_0 \frac{\partial \xi}{\partial t} - S\xi$

$$\frac{\partial^2 \xi}{\partial s^2} = -J_0 \frac{\partial^2 \xi}{\partial s \partial t} - \frac{\partial S}{\partial s}\xi - S\frac{\partial \xi}{\partial s} \Rightarrow \int_0^1 \langle \xi, \frac{\partial^2 \xi}{\partial s^2}\rangle dt =$$

$$-\underbrace{\int_0^1 \langle \xi, J_0 \frac{\partial^2 \xi}{\partial s \partial t} \rangle \, dt} - \int_0^1 \langle \xi, \frac{\partial S}{\partial s} \xi \rangle \, dt - \int_0^1 \langle \xi, S \frac{\partial \xi}{\partial s} \rangle \, dt$$

This term can be computed as follows:

$$0 = \int_{\mathbb{R}/\mathbb{Z}} \frac{d}{dt} \langle \xi, J_0 \frac{\partial \xi}{\partial s} \rangle \, dt \Rightarrow -\int_0^1 \langle \xi, J_0 \frac{\partial^2 \xi}{\partial s \partial t} \rangle \, dt = \int_0^1 \langle \frac{\partial \xi}{\partial t}, J_0 \frac{\partial \xi}{\partial s} \rangle \, dt$$

$$= \int_0^1 \langle J_0 \frac{\partial \xi}{\partial s}, J_0 \frac{\partial \xi}{\partial s} \rangle \, dt + \int_0^1 \langle J_0 S \xi, J_0 \frac{\partial \xi}{\partial s} \rangle \, dt$$

$$= \int_0^1 \left\| \frac{\partial \xi}{\partial s} \right\|^2 dt + \int_0^1 \langle S \xi, \frac{\partial \xi}{\partial s} \rangle \, dt$$

Therefore,

$$f''(s) = \int_0^1 \left\| \frac{\partial \xi}{\partial s}(s,t) \right\|^2 dt + \int_0^1 \left\| \frac{\partial \xi}{\partial s} \right\|^2 dt + \int_0^1 \langle S \xi, \frac{\partial \xi}{\partial s} \rangle \, dt$$

$$- \int_0^1 \langle \xi, \frac{\partial S}{\partial s} \xi \rangle \, dt - \int_0^1 \langle \xi, S \frac{\partial \xi}{\partial s} \rangle \, dt$$

$$= 2 \int_0^1 \left\| \frac{\partial \xi}{\partial s} \right\|^2 dt - \underbrace{\int_0^1 \langle \xi, \frac{\partial S}{\partial s} \xi \rangle \, dt}_{A} + \underbrace{\int_0^1 \langle \xi, (S^T - S) \frac{\partial \xi}{\partial s} \rangle \, dt}_{B}$$

A: $$\int_0^1 \langle \xi, \frac{\partial S}{\partial s} \xi \rangle \, dt \le \left(\int_0^1 \|\xi\|^2 dt \right)^{1/2} \left(\int_0^1 \left\| \frac{\partial S}{\partial s} \xi \right\|^2 dt \right)^{1/2}$$

$$\le \left(\int_0^1 \|\xi\|^2 dt \right)^{1/2} \varepsilon \cdot \left(\int_0^1 \|\xi\|^2 dt \right)^{1/2}$$ $\leftarrow$ Since $\frac{\partial S}{\partial s} \xrightarrow[s \to \pm\infty]{} 0$

$$= \varepsilon \cdot \int_0^1 \|\xi\|^2 dt.$$

B: $$\left| \int_0^1 \langle \xi, (S^T - S) \frac{\partial \xi}{\partial s} \rangle \, dt \right| \le \left(\int_0^1 \|\xi\|^2 dt \right)^{1/2} \varepsilon \left(\int_0^1 \left\| \frac{\partial \xi}{\partial s} \right\|^2 dt \right)^{1/2}$$ $\leftarrow$ $S^T - S \xrightarrow[s \to \pm\infty]{} 0$.

$$\Rightarrow f''(s) \geq 2\int_0^1 \left\|\frac{\partial \xi}{\partial s}\right\|^2 dt - \varepsilon\int_0^1 \|\xi\|^2 dt - \varepsilon\left(\int_0^1 \|\xi\|^2 dt\right)^{1/2}\left(\int_0^1 \left\|\frac{\partial \xi}{\partial s}\right\|^2 dt\right)^{1/2}$$

Note that $\dfrac{\partial \xi}{\partial s} = -J_0\dfrac{\partial \xi}{\partial t} - S\xi = \left(-J_0\dfrac{\partial}{\partial t} - S\right)(\xi)$

still this depends on parameter s.

By SFT-4, $-J_0\frac{\partial}{\partial t} - \lim_{s\to\pm\infty} S$ is Fredholm (from $L^2 \to W^{1,2}$), so when $|s|\gg 1$, $-J_0\frac{\partial}{\partial t} - S$ is also Fredholm $\Rightarrow \exists$ constant $D>0$ s.t.

$$\int_0^1 \left\|\frac{\partial \xi}{\partial s}\right\|^2 dt \geq D\int_0^1 \|\xi\|^2 dt$$

$$\overset{|s|\gg 1}{\Longrightarrow} f''(s) = \left(\int_0^1 \left\|\frac{\partial \xi}{\partial s}\right\|^2 dt\right)^{1/2}\left(2\left(\int_0^1 \left\|\frac{\partial \xi}{\partial s}\right\|^2 dt\right)^{1/2} - \varepsilon\left(\int_0^1 \|\xi\|^2 dt\right)^{1/2}\right) - \varepsilon\int_0^1 \|\xi\|^2 dt$$

$$\geq \sqrt{D}\left(\int_0^1 \|\xi\|^2 dt\right)^{1/2}\left(2\sqrt{D}\left(\int_0^1 \|\xi\|^2 dt\right)^{1/2} - \varepsilon\left(\int_0^1 \|\xi\|^2 dt\right)^{1/2}\right) - \varepsilon\int_0^1 \|\xi\|^2 dt$$

$$= \left(\left(\sqrt{D}\left(2\sqrt{D}-\varepsilon\right)\right) - \varepsilon\right)\int_0^1 \|\xi\|^2 dt$$

$$\overset{\text{choose } \varepsilon \text{ sufficiently small}}{=} \frac{\delta^2}{2}\cdot\int_0^1 \|\xi\|^2 dt = \delta^2 f(s)$$

and set $\delta^2 = 2\left(\sqrt{D}(2\sqrt{D}-\varepsilon)\right)-\varepsilon) > 0$ and $\delta > 0$

we will consider the case when $s\to+\infty$ (consider $e^{\delta s}\ldots$ for $s\to-\infty$).

Now, consider $g(s) = e^{-\delta s}(f'(s) + \delta f(s))$, then

$$g'(s) = e^{-\delta s}\left(-\delta f'(s) - \delta^2 f(s)\right) + e^{-\delta s}\left(f''(s) + \delta f'(s)\right)$$
$$= e^{-\delta s}\left(f''(s) - \delta^2 f(s)\right) \geq 0$$

So $g(s)$ is increasing when s is sufficiently, say $s\in[s_0,\infty)$.

Here are two cases:

<u>Case 1</u> if $\exists\, s_1 \in [s_0, \infty)$ s.t. $g(s_1) > 0$, then for any $s \geq s_1$ we have $g(s) \geq g(s_1)$. Hence for $e^{\delta s} f(s)$, we have

$$\left(e^{\delta s} f(s)\right)' = \delta e^{\delta s} f(s) + e^{\delta s} f'(s)$$
$$= e^{\delta s}\left(f'(s) + \delta f(s)\right) \geq e^{2\delta s} g(s_1) (>0)$$

$\Rightarrow$ for $s \in [s_1, \infty)$, we have

$$e^{\delta s} f(s) - e^{\delta s_1} f(s_1) = \int_{s_1}^{s} \left(e^{\delta \sigma} f(\sigma)\right)' d\sigma \geq g(s_1) \int_{s_1}^{s} e^{2\delta \sigma} d\sigma$$
$$= \frac{g(s_1)}{2\delta}\left(e^{2\delta s} - e^{2\delta s_1}\right)$$

$$\Rightarrow \quad f(s) \geq \underbrace{\frac{g(s_1)}{2\delta} e^{\delta s}}_{\to +\infty} + \underbrace{\frac{\overbrace{\frac{g(s_1)}{2\delta}\left(-e^{2\delta s_1}\right) + e^{\delta s_1} f(s_1)}^{\text{a constant}}}{e^{\delta s}}}_{\to 0}$$

when $s \to +\infty$, $f(s) \to +\infty$ (contradicting to hypothesis)

<u>Case 2</u> Suppose $g(s)$ for all $s \in [s_0, \infty)$ $\left(\Leftrightarrow f'(s) + \delta f(s) \leq 0 \;\; \forall s \in [s_0, \infty)\right)$

$\Rightarrow$ $e^{\delta s} f(s)$ is non-increasing on $[s_0, \infty)$.

$\Rightarrow$ $e^{\delta s} f(s) \leq e^{\delta s_0} f(s_0)$

$\Rightarrow$ $f(s) \leq c \cdot e^{-\delta s}$ when $c = e^{\delta s_0} f(s_0)$ $\square$

<u>Rmk</u> Note that the conclusion in Prop above is not exactly the same as the conclusion ② in Thm above

$$\left\| \frac{\partial u}{\partial s}(s,t) \right\|^2 \quad \text{vs.} \quad \int_0^1 \left\| \frac{\partial u}{\partial s}(s,t) \right\|^2 dt$$

We need the following two lemmas (which are interesting on their own)

Lemma 1 Suppose $\xi : \mathbb{R}\times S^1 \to \mathbb{R}^{2n}$ satisfies $D\xi = 0$, then $\exists\, a > 1$ s.t.

$$\Delta \|\xi\|^2 \geq -a\|\xi\|^2.$$

Suffices to prove $\exists\, a>0$ s.t. the conclusion holds.

Laplacian

Pf. In local coordinate (s,t) on $\mathbb{R}\times S^1$, we have

$$\Delta\langle \xi,\xi\rangle = \frac{\partial^2}{\partial s^2}\langle \xi,\xi\rangle + \frac{\partial^2}{\partial t^2}\langle \xi,\xi\rangle$$

$$= 2\langle \frac{\partial^2}{\partial s^2}\xi,\xi\rangle + 2\langle \frac{\partial \xi}{\partial s},\frac{\partial \xi}{\partial s}\rangle + 2\langle \frac{\partial^2}{\partial t^2}\xi,\xi\rangle + 2\langle \frac{\partial \xi}{\partial t},\frac{\partial \xi}{\partial t}\rangle$$

$$= 2\left(\left\|\frac{\partial \xi}{\partial s}\right\|^2 + \left\|\frac{\partial \xi}{\partial t}\right\|^2 + \langle \Delta\xi,\xi\rangle\right)$$

Observe that $\left(\frac{\partial}{\partial s} - J_0\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s} + J_0\frac{\partial}{\partial t}\right) = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} = \Delta$!

Then $\Delta\xi = \left(\frac{\partial}{\partial s} - J_0\frac{\partial}{\partial t}\right)\left(\frac{\partial \xi}{\partial s} + J_0\frac{\partial \xi}{\partial t}\right)$

Recall $D\xi = 0$ $\Leftrightarrow$ $\frac{\partial \xi}{\partial s} + J_0\frac{\partial \xi}{\partial t} + S\xi = 0$

$$= \left(\frac{\partial}{\partial s} - J_0\frac{\partial}{\partial t}\right)(-S\xi)$$

$$= -\frac{\partial S}{\partial s}\xi - S\frac{\partial \xi}{\partial s} + J_0\frac{\partial S}{\partial t}\xi + J_0 S\frac{\partial \xi}{\partial t}$$

$$\Rightarrow \langle \Delta\xi,\xi\rangle = \langle -\frac{\partial S}{\partial s}\xi,\xi\rangle \underline{- \langle S\frac{\partial \xi}{\partial s},\xi\rangle} + \langle J_0\frac{\partial S}{\partial t}\xi,\xi\rangle + \underline{\langle J_0 S\frac{\partial \xi}{\partial t},\xi\rangle}$$

$$\Rightarrow \frac{1}{2}\Delta\langle \xi,\xi\rangle = \left(\left\|\frac{\partial \xi}{\partial s}\right\|^2 - \langle S\frac{\partial \xi}{\partial s},\xi\rangle\right) + \left(\left\|\frac{\partial \xi}{\partial t}\right\|^2 + \langle J_0 S\frac{\partial \xi}{\partial t},\xi\rangle\right) - \langle \frac{\partial S}{\partial s}\xi,\xi\rangle + \langle J_0\frac{\partial S}{\partial t}\xi,\xi\rangle$$

Cauchy-Schwarz

$|\langle X,Y\rangle| \leq \|X\|\,\|Y\| \Rightarrow$

- $\langle \frac{\partial S}{\partial s}\xi,\xi\rangle \leq C_1\|\xi\|^2$ $\quad C_1$ relates to $\left\|\frac{\partial S}{\partial s}\right\|$
- $\langle J_0\frac{\partial S}{\partial t}\xi,\xi\rangle \geq -C_2\|\xi\|^2$ $\quad C_2$ relates to $\left\|\frac{\partial S}{\partial t}\right\|$

$$\left\|\frac{\partial \xi}{\partial s}\right\|^2 - \langle S\frac{\partial \xi}{\partial t}, \xi\rangle = \left\|\frac{\partial \xi}{\partial s}\right\|^2 - \langle \frac{\partial \xi}{\partial s}, S^T\xi\rangle$$

$$= \left\|\frac{\partial \xi}{\partial s} - \frac{1}{2}S^T\xi\right\|^2 - \frac{1}{4}\|S^T\xi\|^2 \quad \leftarrow \quad \begin{aligned}&\langle X,X\rangle - \langle X,Y\rangle \\ &= \langle X-\tfrac{1}{2}Y, X-\tfrac{1}{2}Y\rangle \\ &\quad -\tfrac{1}{4}\langle Y,Y\rangle\end{aligned}$$

$$\geq -\frac{1}{4}\|S^T\xi\|^2 \geq -C_3\|\xi\|^2 \qquad C_3 \text{ relates to } \|S\|$$

Similarly, $\left\|\frac{\partial \xi}{\partial t}\right\|^2 + \langle J_0 S\frac{\partial \xi}{\partial t}, \xi\rangle \geq -C_4\|\xi\|^2$ when C_4 relates to $\|J_0 S\| = \|S\|$.

All $C_1, C_2, C_3, C_4 \geq 0$, so we get the conclusion. □

<u>Lemma 2</u> $W : B((s_0,t_0),r) \to \mathbb{R}$ a positive C^2-fcn s.t. $\Delta W \geq -b$ for some $b > 0$. Then

$$W(s_0,t_0) \leq \frac{br^2}{8} + \frac{1}{\pi r^2}\int_{B((s_0,t_0),r)} W$$

* Note that if $V : B(0,r) \to \mathbb{R}$ satisfying $\Delta V \geq 0$, then we have a mean value inequality

sub harmonic

$$V(0) \leq \frac{1}{\pi r^2}\int_{B(0,r)} V$$

pf Given such W, define $V : B(0,r) \to \mathbb{R}$

$$V(s,t) = W(s_0+s, t_0+t) + \frac{b}{4}(s^2+t^2)$$

So $V(0,0) = W(s_0,t_0)$ and

$$\Delta V = \Delta W + b \geq 0$$

Then by mean value inequality, we have

$$W(s_0,t_0) = V(0,0) \le \frac{1}{\pi r^2}\int_{B(0,r)} V$$

$$= \frac{1}{\pi r^2}\int_{B((s_0,t_0),r)} W + \frac{b}{4\pi r^2}\int_{B(0,r)} \underline{(s^2+t^2)}\,ds\,dt$$

$$= \frac{b}{4\pi r^2}\int_0^r\int_0^{2\pi} \rho^2\,\rho\,d\rho\,d\theta + \frac{1}{\pi r^2}\int_{B((s_0,t_0),r)} W$$

$$= \frac{b}{\cancel{4\pi r^2}}\, 2\cancel{\pi}\cdot\frac{1}{4}\,\cancel{r^4}^{\,r^2} + \frac{1}{\pi r^2}\int_{B((s_0,t_0),r)} W$$

$$= \frac{br^2}{8} + \frac{1}{\pi r^2}\int_{B((s_0,t_0),r)} W.$$

□

Based on these lemmas, we have the following "ptwise" estimation:

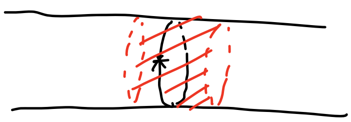
<u>Prop</u> Suppose $\xi:\mathbb{R}\times S^1\to\mathbb{R}^{2n}$ satisfies $D\xi=0$, then $\forall (s_0,t_0)\in\mathbb{R}\times S^1$, we have

$$\|\xi(s_0,t_0)\|^2 \le \frac{8a}{\pi}\int_{B((s_0,t_0),1)}\|\xi(s,t)\|^2\,ds\,dt$$

for some $a\ge 1$.

← Here we can take the radius 1 smaller (to fit into $\mathbb{R}\times S^1$ if needed) then the constant $\frac{8a}{\pi}$ will be changed or rescaled accordingly.

Note that this implies the desired exponential decay of $\|\xi(s_0,t_0)\|$.



(s_0, t) $\quad t\in\mathbb{R}/\mathbb{Z}$

$$\Longrightarrow \int_{s_0-\varepsilon}^{s_0+\varepsilon}\int_0^1\|\xi(s,t)\|^2\,dt\,ds$$

$$\le c\int_{s_0-\varepsilon}^{s_0+\varepsilon} e^{-\delta s}\,ds$$

$$= \frac{c}{-\delta}\left(e^{-\delta(s_0+\varepsilon)} - e^{-\delta(s_0-\varepsilon)}\right)$$

$$= c' e^{-\delta s_0}\cdot O(\varepsilon).$$

apply $\int_0^1\|\xi(s,t)\|^2\,dt \le c e^{-\delta|s|}$ (uniform in s) to a NBH of s_0