Rmk The proof above crucially bases on the fact that there are only finitely many critical pts of F (due to the assumption that F is Morse). Without this assumption, the conclusion in Prop above may fail.

Question: Can we get more accurate description of the convergence behavior near the critical pts ?

Ex (Linear case)

Every Morse fcn $F = \sum \pm x_i$ near critical pt, so consider $\mathbb{R}^n$ and

$$F(\underbrace{x_1, \cdots, x_n}_{x}) = \frac{1}{2} x^T A x$$

where $A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$ where $a_1 \leq \cdots \leq a_k < 0 < a_{k+1} \leq \cdots \leq a_n$.

Then gradient flow $u: \mathbb{R} \to \mathbb{R}^n$ satisfying

$$\begin{aligned} \left(\dot{u}_1(s), \cdots, \dot{u}_n(s)\right) &= (\nabla F)(u(s)) \\ &= \left(a_1 u_1(s), \cdots, a_n u_n(s)\right) \end{aligned}$$

$$\Leftrightarrow \forall i, \quad \dot{u}_i(s) = a_i u_i(s) \Rightarrow u_i(s) = C_i e^{a_i \cdot s}$$

Assume $\lim_{s \to -\infty} u(s) = 0 \in \mathbb{R}^n$, then

- For $a_1, \cdots, a_k < 0$, $u_i(s) \equiv 0$
- For $a_{k+1}, \cdots, a_n > 0$, $u_i(s)$ satisfies $\partial_s^k u_i(s) = \underbrace{C_i a_i^k}_{=: M_k} e^{a_i s}$.

In general, we have the following "exponential decay" phenomenon.

Thm (M, F), $u: \mathbb{R} \to M$ gradient flow (s.t. $\lim_{s \to -\infty} u(s) = p$).

Choose a local coordinate around p s.t. it lies at the origin.

Then $\exists$ a constant $c > 0$, $M_k \in \mathbb{R}$ (for each $k \in \mathbb{N}_{\geq 0}$), s.t.

$\exists\, T \in \mathbb{R}$ with

$$\|\partial_s^k u(s)\| \leq M_k e^{cs} \quad \text{for } s \leq T.$$

An analogue statement holds for positive asymptotics.

Rmk In the example above, $c = \min\{a_{k+1}, \cdots, a_n\} = a_{k+1}$.

Start from the following observation: in local coordinate w.r.t Euclidean metric (where $F(x) = \frac{1}{2}a_1 x_1^2 + \cdots + \frac{1}{2}a_n x_n^2$)

$$\|\nabla F(x)\|^2 = \sum_{i=1}^n (a_i x_i)^2 = \sum_{i=1}^n |a_i||a_i| x_i^2$$

$$\geq \left(\min_{i=1}^n |a_i|\right) 2|F(x)|$$

$$\Rightarrow \underbrace{|F(x)| \leq \frac{1}{2\min\{-a_k, a_{k+1}\}} \cdot \|\nabla F(x)\|^2}_{\text{This is usually called action-energy inequality.}} \quad \text{(assuming again } a_1 \leq \cdots \leq a_k < 0 < a_{k+1} \leq \cdots \leq a_n) \quad Ⓑ$$

Lemma (M, g), $F: M \to \mathbb{R}$, Morse. $p \in \mathrm{Crit}(F)$, then $\exists$ a NBH U of p in M and a constant $k > 0$ s.t. $\forall x \in U$, we have

$$|F(x)| \leq \frac{1}{k} \cdot \|\nabla_g F(x)\|_g^2 \quad \forall x \in U.$$

Pf. It induces the estimation Ⓑ above if we know how this estimation is done in a different metric g.

In general, in a NBH U (of p) in M, two Riem metric g_1, g_2 s.t.

$\exists\, c>0$, $\|\cdot\|_{g_1} \leq c\|\cdot\|_{g_2}$. Then

depending on U, g_1, g_2

$$0 \leq g_1(c^2\nabla_{g_1}F - \nabla_{g_2}F,\ c^2\nabla_{g_1}F - \nabla_{g_2}F)$$

$$= c^4 g_1(\nabla_{g_1}F, \nabla_{g_1}F) - 2c^2 g_1(\nabla_{g_1}F, \nabla_{g_2}F) + g_1(\nabla_{g_2}F, \nabla_{g_2}F)$$

Note that

$$g_1(\nabla_{g_1}F, \nabla_{g_2}F) = dF(\nabla_{g_2}F) = g_2(\nabla_{g_2}F, \nabla_{g_2}F)$$

$$g_1(\nabla_{g_2}F, \nabla_{g_2}F) \leq c^2 g_2(\nabla_{g_2}F, \nabla_{g_2}F)$$

$$\Rightarrow \quad 0 \leq c^4 g_1(\nabla_{g_1}F, \nabla_{g_1}F) - 2c^2 g_2(\nabla_{g_2}F, \nabla_{g_2}F) + c^2 g_2(\nabla_{g_2}F, \nabla_{g_2}F)$$

$$= c^4 g_1(\nabla_{g_1}F, \nabla_{g_1}F) - c^2 g_2(\nabla_{g_2}F, \nabla_{g_2}F)$$

$$\Rightarrow \quad \|\nabla_{g_2}F\|_{g_2}^2 \leq c^2 \|\nabla_{g_1}F\|_{g_1}^2$$

Now, apply this to $g_2 =$ Euclidean metric on $\mathbb{R}^n$; $g_1 = g$ (the given metric on M).

Then
$$|F(x)| \leq \frac{1}{2\min\{-a_k, a_{k+1}\}} \cdot \|\nabla F(x)\|_{g_2} \qquad g_2 = \text{Euclidean}.$$

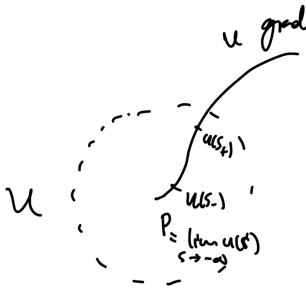
$$\leq \underbrace{\frac{1}{2\min\{-a_k, a_{k+1}\}} \cdot c^2}_{} \cdot \|\nabla_g F(x)\|_g^2 \qquad \forall x \in U$$

depending on U, g_1, g_2

so $$k := \frac{2\min\{-a_k, a_{k+1}\}}{c^2}$$

□

Ex Here is a direct implication of the action-energy inequality above.



Suppose for $s \in [s_-, s_+]$, $F(u(s)) > 0$, then one can estimate the $d_g(u(s_-), u(s_+))$.

one can obtain this by a shift of F s.t. $F(p) = 0$

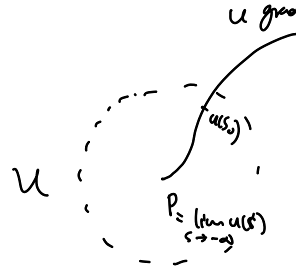
$\leftarrow$ inside $\mathcal{U}$, p is the only pt s.t. $\nabla_g F = 0$.

$$
\begin{aligned}
d_g(u(s_-), u(s_+)) &\leq \int_{s_-}^{s_+} \|\partial_s u(s)\|_g \, ds \\
&= \int_{s_-}^{s_+} \|\nabla_g F(u(s))\|_g \, ds \\
&= \int_{s_-}^{s_+} \frac{\|\nabla_g F(u(s))\|_g^2}{\|\nabla_g F(u(s))\|_g} \, ds \\
&\leq \frac{1}{\sqrt{K}} \int_{s_-}^{s_+} \frac{\|\nabla_g F(u(s))\|_g^2}{\sqrt{F(u(s))}} \, ds \\
&= \frac{1}{\sqrt{K}} \int_{s_-}^{s_+} \frac{dF(\nabla_g F(u(s)))}{\sqrt{F(u(s))}} \, ds \\
&= \frac{1}{\sqrt{K}} \int_{s_-}^{s_+} \frac{dF(\partial_s u(s))}{\sqrt{F(u(s))}} \, ds \\
&= \frac{1}{\sqrt{K}} \int_{s_-}^{s_+} \frac{\partial_s F(u(s))}{\sqrt{F(u(s))}} \, ds \\
&= \frac{2}{\sqrt{K}} \int_{s_-}^{s_+} \partial_s \sqrt{F(u(s))} \, ds = \frac{2}{\sqrt{K}} \left(\sqrt{F(u(s_+))} - \sqrt{F(u(s_-))} \right) \\
&\leq \frac{2}{\sqrt{K}} \sqrt{F(u(s_+))}.
\end{aligned}
$$

apply $\|\nabla_g F(u(s))\|_g \geq \sqrt{K} \cdot \sqrt{F(u(s))}$ for $s \in [s_-, s_+]$

$\|\nabla_g F\|_g^2 = g(\nabla_g F, \nabla_g F) = dF(\nabla_g F)$

Ex Here is another implication of action-energy inequality.



Suppose $\exists\, s_0 \in \mathbb{R}$ s.t. $\forall s \in (-\infty, s_0]$, $F(u(s)) > 0$.

$$F(u(s)) \leq \frac{1}{k}\|\nabla F(u(s))\|_g^2 = \frac{1}{k}\frac{d}{ds}F(u(s))$$

$$\Rightarrow \quad \frac{\frac{d}{ds}F(u(s))}{F(u(s))} \geq k \qquad \text{so} \qquad \frac{d}{ds}\ln F(u(s)) \geq k$$

$$\Rightarrow \quad \int_s^{s_0} \frac{d}{ds}\ln F(u(s))\, ds \geq k\cdot(s_0 - s)$$

$$\Leftrightarrow \quad \ln F(u(s_0)) - \ln F(u(s)) \geq k(s_0-s) \quad \Rightarrow \quad \ln F(u(s)) \leq \ln F(u(s_0)) - k(s_0-s)$$

$$\xrightarrow[\text{exponential}]{\text{take}} \quad F(u(s)) \leq F(u(s_0))\cdot e^{-k(s_0-s)}$$

and this holds for $s \in (-\infty, s_0]$.

Now, back to the proof of Thm above.

Pf. It suffices to prove $\|u(s)\|_g \leq M_0 e^{cs}$ for $s \in (-\infty, T]$.

We can assume $\exists\, T$ s.t. $u(s) \in \mathcal{U}$ (NBH of $p \in \mathrm{Crit}(F)$) for any $s \in (-\infty, T]$, where $\mathcal{U}$ is chosen so that action-energy inequality holds. Up to a shift of F (so that $F(p) = 0$), then both Exs above apply. Then for $[s_-, s_+ \leq s_0] \subset (-\infty, T]$,

we can take $s_0 = T$

$$d_g(u(s_-), u(s_+)) \leq \frac{2}{\sqrt{k}} \sqrt{F(u(s_+))}$$

$$\leq \frac{2}{\sqrt{k}} \sqrt{F(u(s_0))} \cdot e^{-\frac{k(s_0 - s_+)}{2}}$$

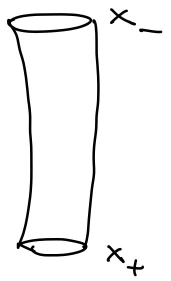
Set $M_0 := \frac{2}{\sqrt{k}} \sqrt{F(u(s_0))}\, e^{-\frac{k s_0}{2}}$ and $c = \frac{k}{2}$. Then set $s_- \to -\infty$,

$$d_g(p, u(s_+)) \leq M_0\, e^{c s_+} \qquad \forall s_+ \in (-\infty, T]$$

Finally, this also implies that $\partial_s u, \partial_s^2 u, \ldots$ converges to 0 exponentially.

3. Exponential decay in Floer theory 1

In Ham Floer theory, the moduli space consists of Ham Floer trajectories



$$u : (\mathbb{R} \times S^1, j_{std}) \longrightarrow (M, \omega, J)$$

$$\bar{\partial}_{H,J}(u) := \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} - \nabla H_t(u) = 0$$

perturbed J-hol curve equation

$$\left(+ \text{ asymptotic condition} \quad \lim_{s \to \pm\infty} u(s,t) = x_\pm(t) \right)$$

we will see from the next thm that $\rightsquigarrow$ this asymptotic condition can be replaced by another equivalent conditions.

Thm ① $E(u) < \infty$ $\Rightarrow$ ② $\exists$ closed Ham orbit $x_\pm$ of (M, ω, J, H)

s.t. $\lim_{s \to \pm\infty} u(s,t) = x_\pm(t)$ and

$\lim_{s\to\pm\infty} \partial_s u(s,t) = 0$ $\rightarrow$ both are uniform in the t-variable

$\Rightarrow$ ③ $\exists\, \delta > 0$ and $C > 0$ s.t. $\|\partial_s u(s,t)\|_{g_J} \leq C e^{-\delta |s|}$

for all $(s,t) \in \mathbb{R} \times S^1 \cong \mathbb{R} \times \mathbb{R}/\mathbb{Z}$

Note that the third one ③ implies the first one ①:

$$E(u) = \int_{S^1} \int_{\mathbb{R}} \|\partial_s u(s,t)\|^2_{g_J}\, ds\, dt \leq C^2 \cdot 1 \cdot \int_{-\infty}^{\infty} e^{-2\delta|s|} ds$$

$$= 2C^2 \int_0^{\infty} e^{-2\delta s} ds$$

$$= \frac{2C^2}{-2\delta} e^{-2\delta s} \Big|_0^{\infty} < \infty.$$

$\Rightarrow$ All three (itemized) statements in this Thm are equivalent!

<u>Exe</u>: ① $\Rightarrow$ ②

We will show ② $\Rightarrow$ ③.

Formulate our setting: For u satisfies $\bar{\partial}_{H,J}(u) = 0$, compute its linearization $D_u : L^p(\mathbb{R}_s \times S^1_t, u^*TM) \longrightarrow W^{1,p}(\mathbb{R} \times S^1, u^*TM)$: $\leftarrow$ could be with even higher regularity k,p

$$D_u(\xi) = \nabla_{\partial_s} \xi + J(u) \nabla_{\partial_t} \xi + (\nabla_\xi J(u))(\partial_t u) - \nabla_\xi \nabla H_t(u)$$

(cf. computation of $\nabla \bar{\partial}_J$ near the end of SFT 2).

One can check $D_u(\partial_s u) = 0$

Under a certain global trivialization of $\begin{matrix} u^*TM \\ \downarrow \\ \mathbb{R}\times S^1 \end{matrix}$, D_u is simplified as

$$D(\xi) = \partial_s \xi + J_0 \partial_t \xi + S\cdot\xi \qquad (\ast)$$

$\xi : \mathbb{R}\times S^1 \to \mathbb{R}^{2n}$

where $S(s,t) : \mathbb{R}\times S^1 \longrightarrow M_{2n\times 2n}(\mathbb{R})$ satisfies $\lim_{s\to\pm\infty} S(s,t) = S_\pm(t)$, $S_\pm(t)$ are symmetric matrices. Moreover, $\lim_{s\to\pm\infty} \frac{\partial S}{\partial s}(s,t) = 0$ (uniform in t).

The proof of ② $\Rightarrow$ ③ lies in the following proposition.

Prop Suppose $\bar\partial_{H,J} u = 0$, and $\xi : \mathbb{R}\times S^1 \to \mathbb{R}^{2n}$ satisfies $D\xi = 0$ and

$$\int_0^1 \|\xi(s,t)\|^2_{g_{J_{std}}} dt \not\to +\infty$$

Euclidean metric

then $\exists\, \delta > 0$ (ind of u) and $C > 0$ s.t. $\int_0^1 \|\xi(s,t)\|^2 dt \leq Ce^{-\delta|s|}$ $\forall s \in \mathbb{R}$.

Pf Consider

$$f(s) = \frac{1}{2}\int_0^1 \|\xi(s,t)\|^2 dt : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$$

Then

$$f'(s) = \int_0^1 \langle \xi(s,t), \frac{\partial \xi}{\partial s}(s,t)\rangle dt$$

$$f''(s) = \int_0^1 \|\frac{\partial \xi}{\partial s}(s,t)\|^2 dt + \int_0^1 \langle \xi(s,t), \frac{\partial^2 \xi}{\partial s^2}(s,t)\rangle dt$$

Due to $(\ast)$, $\frac{\partial \xi}{\partial s} = -J_0 \frac{\partial \xi}{\partial t} - S\xi$

$$\frac{\partial^2 \xi}{\partial s^2} = -J_0 \frac{\partial^2 \xi}{\partial s \partial t} - \frac{\partial S}{\partial s}\xi - S\frac{\partial \xi}{\partial s} \Rightarrow \int_0^1 \langle \xi, \frac{\partial^2 \xi}{\partial s^2}\rangle dt =$$