– Consider $L < \frac{r_0}{6}$. We claim $E(u)<\hbar$ implies that $\exists\ B(p, \frac{r_0}{3}) \subset M$, s.t.

$$B(p, \tfrac{r_0}{3}) \supset u(\Sigma)$$

To prove this claim, we divide into two cases.

<u>Case 1</u> $\partial\Sigma = \emptyset$. (In this case, we don't need cond. $L<\frac{r_0}{6}$)

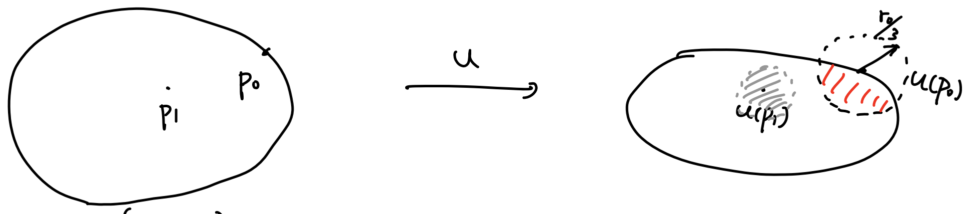
Then the assumption in monotonicity lemma holds automatically. If none of $B(p, \frac{r_0}{3}) \subset M$ contains $u(\Sigma)$, then

$$\hbar > E(u) = \text{Area}(u) > \text{Area}(u(\Sigma) \cap B(p, r)) \geq c r^2$$

for any $r < \frac{r_0}{3}$. Then up to ε, we get

$$c \cdot \left(\frac{r_0}{6}\right)^2 > c\left(\frac{r_0}{3}\right)^2 \quad \rightarrow \leftarrow.$$

<u>Case 2</u> $\partial\Sigma \neq \emptyset$. Again, suppose the conclusion does NOT hold.



$(\Sigma, \partial\Sigma)$

$$u(\Sigma) \not\subset B(u(p_0), \tfrac{r_0}{3}) \iff u(\Sigma) \setminus (u(\Sigma) \cap B(u(p_0), \tfrac{r_0}{3})) \text{ is non-empty}$$

$\Rightarrow$ for $p_1 \in \Sigma$ with $\text{dist}(u(p_1), u(p_0)) \geq \frac{r_0}{3}$, we have

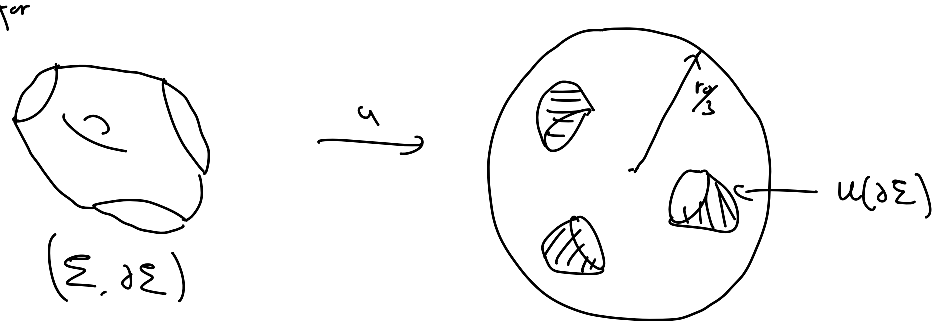
$$\text{dist}(u(p_1), u(\partial\Sigma)) \geq \frac{r_0}{6}$$

(b/c otherwise $\text{dist}(u(p_1), u(p_0)) \leq \text{dist}(u(p_1), u(p_*)) + \text{dist}(u(p_*), u(p_0))$, $p_* \in \partial\Sigma$, $< \frac{r_0}{6} + \frac{r_0}{6} \leftarrow$ condition $L < \frac{r_0}{6}$)

$$\Rightarrow B(u(p_1), \tfrac{r_0}{6}) \cap u(\partial\Sigma) = \emptyset$$

$$\Rightarrow E(u) \geq \text{Area}(u(\Sigma) \cap B(u(p_1), \tfrac{r_0}{6})) \geq c\left(\frac{r_0}{6}\right)^2 \rightarrow \leftarrow.$$

Now, for



$v : (\coprod_i \mathbb{D}^2, j) \to (M, \omega, J)$

s.t $v|_{\coprod \partial \mathbb{D}^2} = u(\partial\Sigma)$

b/c $\omega|_{B(p, \frac{r_0}{3})}$ = exact

$$E(u) = \int_\Sigma u^*\omega = \int_{\coprod \mathbb{D}^2} v^*\omega = \sum_i \int_{\mathbb{D}^2} v_i^*\omega \le \sum_i \frac{1}{4C} \text{length}^2_{g_J}(v_i(\partial\mathbb{D}^2))$$

$v_i = v|_{\mathbb{D}^2}$; isoperimetric inequality.

$$= \frac{1}{4C} \text{length}^2_{g_J}(u(\partial\Sigma)).$$

Finally, one can approximate $\partial\Sigma = \emptyset$ case by $\Sigma \setminus$ small disk. $\square$

Here is another example how to apply the monotonicity lemma.

$\underline{\text{Ex}}$ $(\Sigma, j) = (\{z \in \mathbb{C} \mid r < |z| < R\}, j)$ for some r and R.

the constant in monotonicity lemma

$u : (\Sigma, j) \to (M, \omega, J)$ J-hol and $E(u) < C\varepsilon^2$

where $\varepsilon < r_0/3$ and $L(u(\circ_r)), L(u(\circ_R)) < \varepsilon$.

Then

$$\text{diam}(u(\Sigma)) = \sup\{\text{dist}(u(p), u(p')) \mid p, p' \in \Sigma\} < 5\varepsilon.$$

To prove this statement, we only need to show

$\text{diam}(u(\Sigma)) \ge 5\varepsilon$ + $L(u(\circ_r)), L(u(\circ_R)) < \varepsilon \Rightarrow$ $\exists p \in \Sigma$ s.t $\text{dist}(u(p), u(\partial\Sigma)) > \varepsilon$.

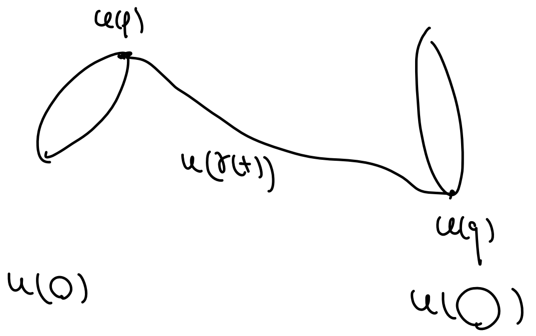
Then $u(\partial\Sigma) \cap B(p,\varepsilon) = \emptyset$ and then monotonicity lemma says

$$E(u) \geq C\varepsilon^2 \quad \rightarrow \Leftarrow.$$

Here, we argue in two cases

<u>Case 1</u> assume $\mathrm{dist}(u(\circ), u(\bigcirc)) \leq 2\varepsilon$. (Good Exercise)

<u>Case 2</u> assume $\mathrm{dist}(u(\circ), u(\bigcirc)) > 2\varepsilon$.



$p \in \circ$, $q \in \bigcirc$
s.t. $\mathrm{dist}(u(p), u(q)) = \mathrm{dist}(u(\circ), u(\bigcirc))$

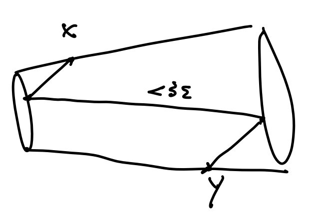
Then $t \mapsto \mathrm{dist}(u(\gamma(t)), u(\circ))$ is a continuous fcn that

$0 \mapsto 0$ and $1 \rightarrow \mathrm{dist}(u(\circ), u(\bigcirc))$.

By IVT, $\exists\, t_0 \in (0,1)$ s.t. $\mathrm{dist}(\gamma(t_0), u(\circ)) = \frac{1}{2}\mathrm{dist}(u(\circ), u(\bigcirc)) (>\varepsilon)$

triangle inequality
$\Rightarrow \mathrm{dist}(\gamma(t_0), u(\bigcirc)) \geq \frac{1}{2}\mathrm{dist}(u(\circ), u(\bigcirc)) > \varepsilon$.

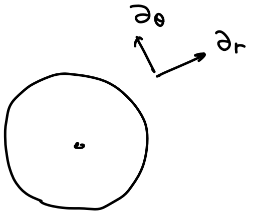
(Hint for case 1: show that $\mathrm{dist}(u(\circ), u(\bigcirc)) \leq 2\varepsilon \Rightarrow \forall\, u(p), u(q) \in u(\partial\Sigma)$
have $\mathrm{dist}(u(p), u(q)) < 3\varepsilon$. Then



$\mathrm{dist}(u(x), u(y)) = \mathrm{diam}(u(\Sigma)) \geq 5\varepsilon$.
$\Rightarrow$ either $\mathrm{dist}(u(x), u(\partial\Sigma)) > \varepsilon$
or $\mathrm{dist}(u(y), u(\partial\Sigma)) > \varepsilon$.)

Now, back to the proof of Removal of singularities.

Choose a global polar coordinate (r, θ) on $\overset{B(0,1)}{\mathbb{D}}\setminus\{0\}$, where



$$j\,\partial_\theta = -r\,\partial_r \quad \left(\Leftrightarrow \partial_r = j\left(-\tfrac{1}{r}\partial_\theta\right)\right)$$

Then

$$\begin{aligned}
(u^*\omega)(\partial_r, \partial_\theta) &= \omega(du(\partial_r), du(\partial_\theta)) \\
&= \omega\left(-\tfrac{1}{r}\, du \cdot j(\partial_\theta), du(\partial_\theta)\right) \\
&= \tfrac{1}{r}\,\omega(-J \cdot du(\partial_\theta), du(\partial_\theta)) \\
&= \tfrac{1}{r}\,\omega\left(\tfrac{\partial u}{\partial \theta}, J\tfrac{\partial u}{\partial \theta}\right) = \tfrac{1}{r}\left|\tfrac{\partial u}{\partial \theta}\right|^2_{g_J}
\end{aligned}$$

$$\Rightarrow E(u) = \int_{\mathbb{D}\setminus\{0\}} u^*\omega = \int_0^1 \int_0^{2\pi} \frac{1}{r}\left|\frac{\partial u}{\partial \theta}\right|^2_{g_J} d\theta\, dr$$

To deal with the behavior near 0, let's introduce some notations

$$D^*(r) = B(0,r)\setminus\{0\} \quad \text{and} \quad A(r,R) = \{z \in \mathbb{C} \mid r \le |z| \le R\}$$
$$D(r) = B(0,r)$$

$$\alpha(t) = \int_0^t \int_0^{2\pi} \frac{1}{r}\left|\frac{\partial u}{\partial \theta}\right|^2_{g_J} d\theta\, dr \quad \left(= E(u(D(t)))\right)$$

$$\lambda(t) = \mathrm{length}_{g_J}(u(\partial D(t))) \quad \left(= \int_0^{2\pi}\left|\frac{\partial u}{\partial \theta}\right|_{g_J} d\theta\right)$$

Recall $\displaystyle \frac{d}{dt}\int_{g(t)}^{f(t)} h(r)\,dr = h(f(t))\cdot f'(t) - h(g(t))\cdot g'(t).$

$$\Rightarrow \alpha'(t) = \frac{1}{t}\int_0^{2\pi}\left|\frac{\partial u}{\partial \theta}\right|^2_{g_J} d\theta$$

$$\Rightarrow \quad \lambda(t)^2 = \left(\int_0^{2\pi} \left| \frac{\partial u}{\partial \theta} \right|_{g_J} d\theta \right)^2 \leq 2\pi \cdot \underbrace{\int_0^{2\pi} \left| \frac{\partial u}{\partial \theta} \right|_{g_J}^2 d\theta}_{t \cdot \alpha'(t)}$$

$$\Rightarrow \quad \lambda(t)^2 \leq 2\pi t \, \alpha'(t)$$

$$\Rightarrow \quad \alpha'(t) \geq \frac{1}{2\pi t} \lambda(t)^2$$

Then for any $0 < \eta < \rho < 1$, we have

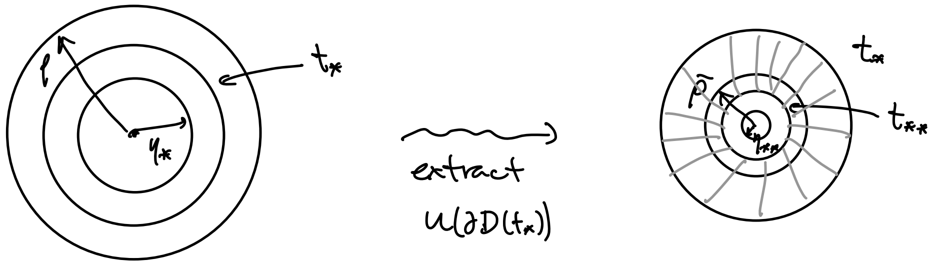
$$\alpha(\rho) \geq \alpha(\rho) - \alpha(\eta) \geq \int_\eta^\rho \frac{1}{2\pi t} \lambda(t)^2 dt$$

$$\geq \min_{t \in [\eta, \rho]} \lambda(t)^2 \cdot \frac{1}{2\pi} \int_\eta^\rho \frac{1}{t} dt$$

$$= \min_{t \in [\eta, \rho]} \lambda(t)^2 \cdot \frac{1}{2\pi} \cdot \ln\left(\frac{\rho}{\eta}\right)$$

Now, $E(u) < \infty$ implies that when $\rho \to 0$, $\alpha(\rho) \to 0$. Therefore, for any fixed ρ s.t. $\alpha(\rho) < C\varepsilon^2$, since $\ln\left(\frac{\rho}{\eta}\right) \to \infty$ as $\eta \to 0^+$, we have $\exists \, \eta_* \in (0, \rho]$ s.t.

$$\varepsilon > \min_{t \in [\eta_*, \rho]} \lambda(t) = \lambda(t_*) \text{ for some } t_* \in [\eta_*, \rho].$$

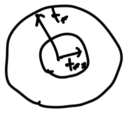


Apply the argument one more time for interval $[\eta_{**}, \overset{<t_*}{\tilde{\rho}}]$, $\exists\, t_{**} \in [\eta_{**}, \tilde{\rho}]$

This $\tilde{\rho}$ can be taken arbitrarily small (close to 0)

s.t

$$\varepsilon > \min_{t \in [\eta_{**}, \tilde{\rho}]} \lambda(t) = \lambda(t_{**})$$

Then by Ex above applied to  , we have.

$$\operatorname{diam}\big(u(A(t_{**}, t_*))\big) < 5\varepsilon.$$

$$\Rightarrow \operatorname{diam}\big(u(A(\tilde{\rho}, t_*))\big) < 5\varepsilon \quad \text{for any } \tilde{\rho} \in (t_{**}, t_*].$$

$$\overset{\tilde{\rho} \to 0}{\Rightarrow} \operatorname{diam}\big(u(D^*(t_*))\big) < 5\varepsilon.$$

For any seq $z_n \in D^*(t_*)$ where $z_n \to 0$, $u(z_n)$ is a Cauchy seq

Since M is cpt (so it is complete), $\exists$ a limit pt $u_0 \in M$.

Then define $u(0) := u_0$.

To sum up, we have shown that $\forall \varepsilon > 0$, $\exists$ a NBH of 0, say $D(t_*)$ s.t. $\forall\, u(z) \in D(t_*)$, we have $\operatorname{dist}(u(z), u_0) < 5\varepsilon$.

$\Rightarrow u$ is continuous at pt 0. $\square$