4. Energy revisit

$(M^{2n-1}, (\omega, \lambda))$ stable Ham str.

$\mathcal{J}((\omega,\lambda)) = \{$ a.c.s on $\mathbb{R}\times M$ satisfying $(*)\}$

$T_\varepsilon = \{ \varphi : \mathbb{R} \to (-\varepsilon, \varepsilon) \mid \varphi' > 0 \}$ for any fixed $\varepsilon > 0$.
test function

in fact, it should be $\dot{\Sigma} = \Sigma \setminus \{pts\}$

Then for any J-hol curve $u : (\Sigma, j) \to (\mathbb{R}\times M, J)$, one can define its energy

$$E_\varepsilon(u) := \sup_{\varphi \in T_\varepsilon} \int_\Sigma u^* \omega_\varphi$$

$\leftarrow$ so $E_\varepsilon(u)$ is independent of the test function.

If $J \in \mathcal{J}((\omega,\lambda))$, then $E_\varepsilon(u) \geq 0$ and $E_\varepsilon(u) = 0$ iff u is constant.

Rmk Def of $E_\varepsilon(u)$ is good since "trivial" cylinder admits small energy (cf. Section 1)

Question What's the relation between $E_\varepsilon(u)$ and $E_\delta(u)$?

In general, let's consider

$$T_{(a,b)} := \{ \varphi : \mathbb{R} \to (a,b) \subset (-\varepsilon, \varepsilon) \mid \varphi' > 0 \}$$

and $$E_{(a,b)}(u) := \sup_{\varphi \in T_{(a,b)}} \int u^* \omega_\varphi$$

Prop. $\exists\, C(a,b,\varepsilon)$, independent of u, s.t. $C E_\varepsilon(u) \leq E_{(a,b)}(u) \leq E_\varepsilon(u)$.

Pf. The inequality $E_{(a,b)}(u) \leq E_\varepsilon(u)$ is trivial b/c $T_{(a,b)} \subset T_\varepsilon$.

Start from the tamed condition $\omega(v, Jv) > 0$ for any $v \in \ker(\lambda)$

This extends to $T(\mathbb{R}\times M)$ b/c $\omega(\partial_t, -) = \omega(R, -) = 0$.

depending on $\omega, d\lambda, \varepsilon$

$\Rightarrow \exists\ c > 1$ s.t.

b/c tamed is an open condition

$$\min\{c-1, 1-\tfrac{1}{c}\}\,\omega(X, JX) > |k\,d\lambda(X, JX)| \quad \forall X \in T(\mathbb{R}\times M),\ k \in (-\varepsilon, \varepsilon)$$

(Here, we assume ε sufficiently small)

$$\Rightarrow \quad \frac{1}{c}(\omega + k\,d\lambda)(X, JX) \le \omega(X, JX) \le c(\omega + k\,d\lambda)(X, JX)$$

Suppose $\varphi \in T_\varepsilon$, for any constant $\delta \in (0, b-a]$, define

$$\tilde{\varphi}(r) := \frac{\delta}{2\varepsilon}\varphi(r) + \frac{a+b}{2} \in (a, b)$$

$\frac{\delta}{2\varepsilon}\cdot\varepsilon + \frac{a+b}{2} \le b$

$\frac{\delta}{2\varepsilon}(-\varepsilon) + \frac{a+b}{2} \ge a$

So $\tilde{\varphi} \in T_{(a,b)}$. Then

$$\int_\Sigma u^*\omega_\varphi = \int_\Sigma u^*(\omega + \varphi(r)d\lambda + \varphi'(r)\,dr\wedge\lambda)$$

$$= \int_\Sigma u^*(\omega + \varphi(r)d\lambda) + \int_\Sigma \varphi'(r)\,dr\wedge\lambda$$

$$\le c^2\int u^*(\omega + \tilde{\varphi}(r)d\lambda) + \frac{2\varepsilon}{\delta}\int_\Sigma \tilde{\varphi}'(r)\,dr\wedge\lambda$$

comparison between c ($\exists$ above) and $\frac{2\varepsilon}{b-a}$ so the desired constant in conclusion is $\min\{\frac{1}{c^2}, \frac{b-a}{2\varepsilon}\}$

If $c^2 \ge \frac{2\varepsilon}{b-a}$, then choose $\delta = \frac{2\varepsilon}{c^2} \in (0, b-a]$, then

$$\int_\Sigma u^*\omega_\varphi \le c^2\int u^*\omega_{\tilde{\varphi}} \le c^2 E_{(a,b)}(u)$$

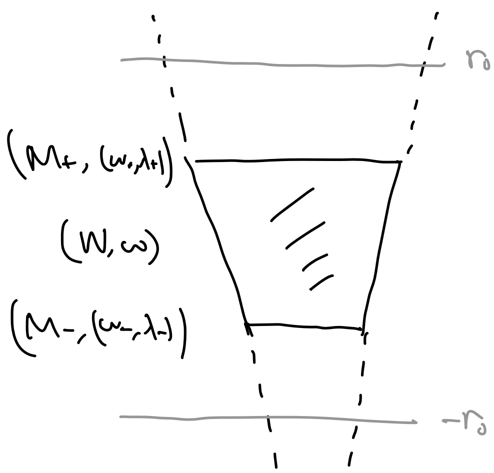
$$\Rightarrow \quad \frac{1}{c^2}E_\varepsilon(u) \le E_{(a,b)}(u)$$

If $c^2 < \frac{2\varepsilon}{b-a}$, then choose $\delta = b-a$, then

$$\int_\Sigma u^*\omega_\varphi \le \frac{2\varepsilon}{b-a}\int_\Sigma u^*\omega_{\tilde{\varphi}} \le \frac{2\varepsilon}{b-a}E_{(a,b)}(u) \Rightarrow \text{conclusion} \quad \square$$

Rmk Prop above shows that whether $E_\varepsilon(u)$ is bounded is independent of ε as long as it is sufficiently small.

Question. How to generalize the energy $E_\varepsilon(u)$ to symplectic cobordism setting (+ completion)?



Fix $\psi \in \mathcal{T}_\varepsilon^+$ ($\psi(r)=r$ near 0.) and $r_0 \geq 0$, denote

$\mathcal{J}(\omega_\psi, r_0, (\omega_+,\lambda_+), (\omega_-,\lambda_-))$

$= \{ J$ a.c.s on $\widehat{W}$ |
$J|_{[r_0,\infty)\times M_+} \in \mathcal{J}((\omega_+,\lambda_+))$
$J|_{(-\infty,-r_0]\times M_-} \in \mathcal{J}((\omega_-,\lambda_-))$
$J|_{([-r_0,0]\times M_-)\cup_{M_-} W \cup_{M_+}([0,r_0]\times M_+)}$ is ω_ψ-compatible$\}$

Then define

$$E_{\psi,\varepsilon,r_0}(u) := \sup_{\varphi \in \mathcal{T}_{\psi,\varepsilon,r_0}} \int_\Sigma u^*\omega_\varphi$$

b/c by the same argument as above. if $(\omega_\pm,\lambda_\pm)$ are stable, then J chosen here is ω_φ-tamed for any $\varphi \in \mathcal{T}_{\psi,\varepsilon,r_0}$

where $\mathcal{T}_{\psi,\varepsilon,r_0} := \{ \varphi \in \mathcal{T}_\varepsilon \mid \varphi \equiv \psi$ on $[-r_0, r_0]\}$.

5. Removal of singularities

In practical case, the domain of our J-hol map $(\Sigma, j) \xrightarrow{u} (\widehat{W}, \omega_\varphi)$ will be punctured Riem surface $\dot{\Sigma} = \Sigma \setminus \Gamma$.

a finite set of pts on Σ.

Near each pt $p \in \Gamma$, locally one can view the NBH in three different way:

(i) $\mathbb{D}\setminus\{0\}$

(ii) $Z_+ := [0,\infty) \times S^1$ ($\xrightarrow[\sim]{\text{bihol}}$ $\mathbb{D}\setminus\{0\}$ by $(s,t) \to e^{-2\pi(s+it)}$)

(iii) $Z_- := (-\infty, 0] \times S^1$ ($\xrightarrow[\sim]{\text{bihol}}$ $\mathbb{D}\setminus\{0\}$ by $(s,t) \to e^{2\pi(s+it)}$)

Though topologically (i) – (iii) are the same, when energy is involved, (i) is fundamentally different from (ii) and (iii)

Thm Given $\overset{\text{cpt}}{}$ (X, ω, J) where J is ω-tamed, $u: (\mathbb{D}\setminus\{0\}, j) \to (X, J)$ a smooth J-hol curve that $\int_{\mathbb{D}\setminus\{0\}} u^*\omega < \infty$. Then u admits a continuous extension to $\mathbb{D}$.

this is called removal of singularity (at $0 \in \mathbb{D}$).

Rmk By adding further regularity (where u is continuous on $\mathbb{D}$ and smooth on $\mathbb{D}\setminus\{0\}$) one can prove that u extends to 0 in a smooth way). cf Lemma 9.7 in [Wen]
cf. Last Thm in SFT-3.

Ex consider $u: \mathbb{D}\setminus\{0\} \to (\mathbb{C}, \sqrt{-1})$ $u(z) = \frac{1}{z}$ (holomorphic). Then obviously it can not be extended to $\{0\} \in \mathbb{D}$. Also, note that

$$\int_{\mathbb{D}\setminus\{0\}} u^*\omega_{std} = \int_0^{2\pi}\int_0^1 \left|\frac{\partial u}{\partial \rho}\right|^2_{g_J} \rho\, d\rho \wedge d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{\rho^3}\, d\rho \wedge d\theta = \pi \cdot \frac{-1}{\rho^2}\Big|_0^1 = \pi\left(-1 + \frac{1}{0^+}\right) = +\infty.$$

$$\frac{\partial u}{\partial \rho} = \frac{\partial(\frac{1}{\rho}e^{-\sqrt{-1}\theta})}{\partial \rho} = -\frac{1}{\rho^2}e^{-\sqrt{-1}\theta}$$

Thm above says that under the "local energy bounded" condition, puncture pts can be neglected.

The proof of thm relies on the following well-known result (in minimal surface theory).

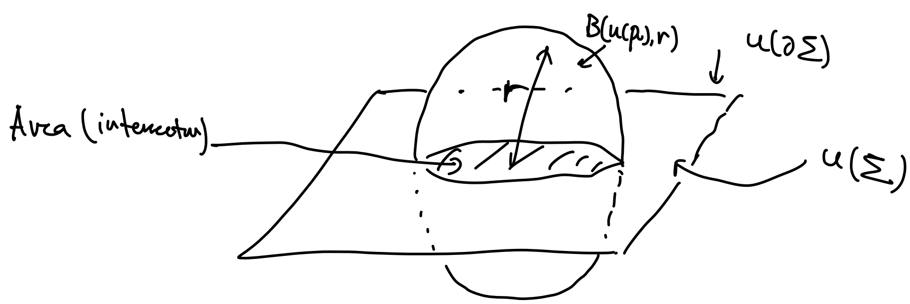
FACT (monotonicity lemma) Given cpt (X, ω, J) where J is ω-tamed, and $r_0 := \mathrm{inj}_{M, g_J} (>0)$ the injective radius, a non-constant J-hol curve $u: (\Sigma, j) \longrightarrow (X, J)$, suppose $p_0 \in \Sigma$, $0 < r < r_0/3$, and

connected cmpt with b/d

$$u(\partial\Sigma) \cap B(u(p_0), r) = \emptyset$$

ball centered at $u(p_0)$ with radius r

Then $\mathrm{Area}(u(\Sigma) \cap B(u(p_0), r)) \geq C r^2$ for a constant C ind of u.



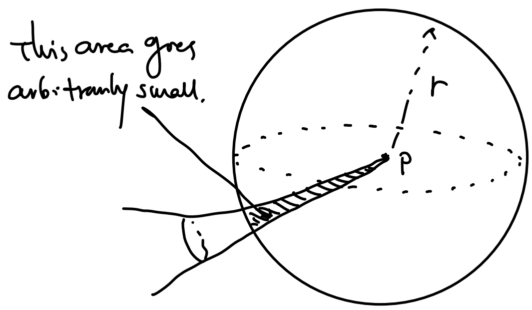
Ex $(M, \omega, J) = (\mathbb{R}^{2n}, \omega_{std}, J = J_0)$ (then $r_0 = \infty$). One can check that C in monotonicity lemma above $= \pi$.

$\Rightarrow$ any J_0-hol curve in $(\mathbb{R}^{2n}, \omega_{std}, J_0)$ passing through 0 in $B(0, r)$ must have area at least πr^2

with b/d not inside $B(0,r)$

Rmk • Monotonicity lemma says a J-hol curve must use up at least a certain amount of area (= energy) for every ball whose center it passes through.

• Here is an example (not J-hol, hence) that violates the conclusion of monotonicity lemma.



The following "quantum type" result shows how one applies this monotonicity lemma.

Prop $\exists \hbar > 0$ s.t. $u : (\Sigma, j) \longrightarrow (M, \omega, J)$ J-hol and Σ is cpt.

cpt

If $E(u) < \hbar$, then $E(u) \leq \frac{1}{4c} \text{length}^2_{g_J}(u(\partial\Sigma))$. ← this c is the one in monotonicity lemma

In particular, if $\partial\Sigma = \emptyset$, then any non constant J-hol map must have energy at least $\hbar$. ← in other words, energy does not change continuously.

Pf the second conclusion is trivial since $E(u) = 0 \Leftrightarrow u$ is constant.

Choose $\hbar = \min\{\frac{1}{4c}, c\} \cdot \left(\frac{r_0}{6}\right)^2$ and denote $L = \text{length}_{g_J}(u(\partial\Sigma))$

- If $L \geq \frac{r_0}{6}$, then $E(u) < \hbar$ implies that

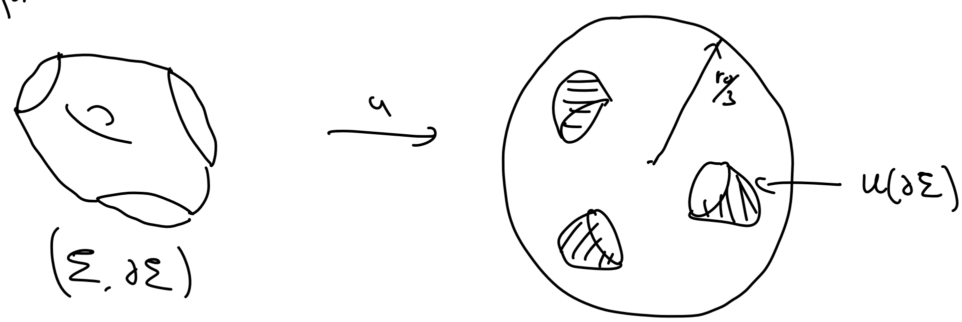
$$E(u) \leq \frac{1}{4c}\left(\frac{r_0}{6}\right)^2 = \frac{1}{4c} \cdot L^2 = \frac{1}{4c} \text{length}^2_{g_J}(\partial u(\Sigma)).$$

- Consider $L < \frac{r_0}{6}$. We claim $E(u) < \hbar$ implies that $\exists\ B(p, \frac{r_0}{3}) \subset M$, s.t.

$$B(p, \tfrac{r_0}{3}) \supset u(\Sigma) \qquad (**)$$

Assume this (the proof is by monotonicity lemma and will be presented NEXT TIME), then we can finish the proof of this prop.

Now, for



$v : (\coprod_i \mathbb{D}^2, j) \longrightarrow (M, \omega, J)$

s.t. $v|_{\coprod \partial \mathbb{D}^2} = u(\partial\Sigma)$

b/c $\omega|_{B(p, \frac{r_0}{3})} = \text{exact}$

$$E(u) = \int_\Sigma u^*\omega = \int_{\coprod \mathbb{D}^2} v^*\omega = \sum_i \int_{\mathbb{D}^2} \underbrace{v_i}_{v|_{\mathbb{D}^2}}{}^*\omega \underset{\text{isoperimetric inequality.}}{\leq} \sum_i \frac{1}{4c} \text{length}^2_{g_J}(v_i(\partial\mathbb{D}^2))$$

$$= \frac{1}{4c} \text{length}^2_{g_J}(u(\partial\Sigma)).$$

Finally, one can approximate $\partial\Sigma = \emptyset$ case by $\Sigma \setminus$ small disk. $\square$

modulo the proof (**) above.