by a Ham fcn $H : [0,1] \times X \to \mathbb{R}$ (s.t. $\varphi_H^1 = \varphi$), then $\exists$ a global trivialization (as a surjective)

$$\Phi_H : X \times S^1(\theta) \longrightarrow Y_\varphi$$
$$(x, \theta) \longrightarrow [((\varphi_H^\theta)^{-1}(x), \theta)]$$

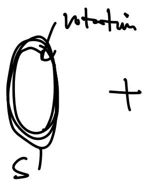
Then
$$\Phi_H^* \lambda = \Phi_H^*(\pi^* d\theta) = (\pi \circ \Phi_H)^* d\theta = \pi^* d\theta$$
$$\Phi_H^* \omega_\varphi = \Phi_H^*(\pi_X^* \omega) = (\pi_X \circ \Phi_H)^* \omega = \omega - dH \wedge d\theta.$$

$\Rightarrow$ $(\Phi_H^* \omega_\varphi, \Phi_H^* \lambda)$ is a framed Ham str on $S^1(\theta) \times X$.

One can also check that the Reeb v.f. of $(\Phi_H^* \omega_\varphi, \Phi_H^* \lambda)$ is

$$(\Phi_H^{-1})_*(R) = \frac{\partial}{\partial \theta} + X_H$$

(b/c
$$(\Phi_H^* \omega_\varphi)((\Phi_H^{-1})_*(R), -) = (\omega - dH \wedge d\theta)\left(\frac{\partial}{\partial \theta} + X_H, -\right)$$
$$= \omega(X_H, -) + dH(-) - \overset{0}{\cancel{dH(X_H)}}\, d\theta = 0$$

$\Rightarrow$ flow of $(\Phi_H^{-1})_* R$ is 

$+$ Ham flow of $H$

$\Rightarrow$ closed "Reeb" orbits on $S^1 \times X$ $\simeq$ closed Hamiltonian orbits after integer time period. $\leftarrow$ In Ham Floer homology theory, one usually considers 1-periodic closed Ham orbit.

In this way, one can consider "all" Ham periodic orbits in a uniform framework.

**Def** A framing $\lambda$ in $(\omega,\lambda)$ on $M^{2n-1}$ is called <u>stable</u> if

$$d\lambda(R_{(\omega,\lambda)}, \cdot) \equiv 0 \qquad (*)$$

where $R_{(\omega,\lambda)}$ is the Reeb v.f. of $(\omega,\lambda)$.

Note that since $\omega(R,-)\equiv 0$, we know $(*) \Leftrightarrow d\lambda = f\cdot\omega$ for some function $f: M \to \mathbb{R}$. Since $f$ could be zero, so $(*) \Leftrightarrow \ker(\omega)\subset \ker(d\lambda)$.

**Ex** Contact mfd $(\omega,\lambda)=(d\alpha,\alpha) \Rightarrow$ stable

mapping torus $(\omega_\varphi,\lambda) \Rightarrow$ stable

**Rmk** Recall that the flow of $R_{(\omega,\lambda)}$ preserves $\omega$ (for $(\omega,\lambda)$)

Here, for stable Ham str, we have

$$L_{R_{(\omega,\lambda)}}\lambda = d\underbrace{\iota_{R_{(\omega,\lambda)}}\lambda}_{\equiv 1} + \iota_{R_{(\omega,\lambda)}}d\lambda = 0+0=0$$

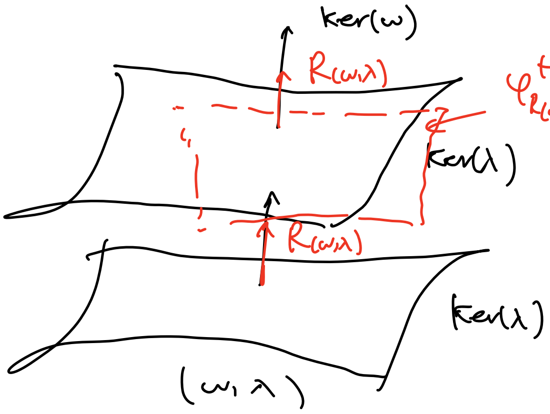
$\Rightarrow$ the flow of $R_{(\omega,\lambda)}$ also preserves $\lambda$.

This serves as one benefit of considering stable $(\omega,\lambda)$.

$\Downarrow$

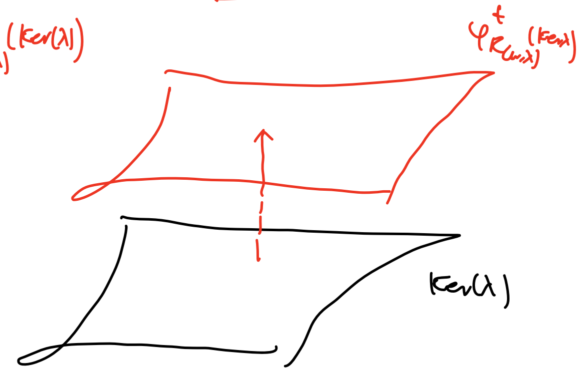
$\varphi^t_{R_{(\omega,\lambda)}}$ preserves $\ker(\lambda)$

Comparison:




$(\omega,\lambda)$

framed Ham str.




$(\omega,\lambda)$ stable.

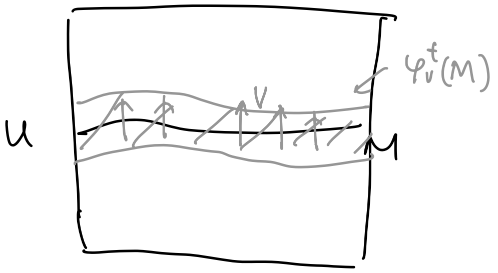
## 3. Symplectic cobordism

<u>Exe</u>. $(X^{2n}, \omega)$ symp mfd, $M^{2n-1} \subset X^{2n}$ cpt hypersurface. Then TFAE:

(1) $\omega|_M$ admits a framing $\lambda$ s.t. $(\omega|_M, \lambda)$ is stable on $M$.

$\lambda := \omega(V,-)|_{TM}$

(2) $\exists$ a NBH $\mathcal{U}$ of $M$ in $X$ that admits a v.f. $V \pitchfork M$, its flow $\varphi_V^r$ satisfies $\varphi_V^r(M) \underset{\text{diffeo}}{\simeq} M$ for $t \in (-\varepsilon, \varepsilon)$ and some $\varepsilon > 0$, and $(\varphi_V^r)_*: \ker(\omega|_M) \circlearrowleft$.




In other words, NBH $\mathcal{U}$ of $M$ in $X$ is foliated by copies of $M$ (up to diffeo) in an "orthogonal" way.

$\Rightarrow$ via $\varphi_V^r$, one can identify this NBH $\mathcal{U}$ of $M$ with $(-\varepsilon, \varepsilon) \times M$

$$\begin{aligned} (-\varepsilon, \varepsilon) \times M &\xrightarrow[\sim]{\Phi_V} \mathcal{U} \\ (r, \ x) &\longrightarrow \varphi_V^r(x) \end{aligned} \qquad \text{where} \quad \begin{aligned} \{0\} \times M &\simeq M \\ \partial_r &\simeq V \end{aligned}$$

- Then study the closed 2-form $\Phi_V^* \omega$ near $M$.

Observe that along $\{0\} \times M$, we have

$$(\Phi_V^* \omega)(a, b) = \omega|_{TM}((\varphi_V^0)_*(a), (\varphi_V^0)_*(b)) = \omega|_{TM}(a, b)$$

$$(\Phi_V^* \omega)(\partial_r, -) = \omega|_{TM}(V, -) =: \lambda \leftarrow \text{fixed once } \omega, V \text{ are fixed.}$$

($a, b \in TM$)

$\Rightarrow$ $\Phi_V^* \omega = \omega|_{TM} + dr \wedge \lambda$ on $\{0\} \times M$

this should be understood as the pullback $\pi_M^* \omega|_{TM}$ for projection $\pi_M: (-\varepsilon, \varepsilon) \times M \to M$

$\Rightarrow$ consider $\omega|_{TM} + d(r\lambda)$ in a NBH of $M$.

When $|t|$ is sufficiently small, it is also a symplectic str.

**|r| sufficiently small - since here is an extra term r d\lambda, where d\lambda = f \omega (by the stability of \lambda); therefore if |r| is sufficiently small, this new 2-form serves as a small perturbation of \omega + dr \wedge \lambda, which is also non-degenerate, implying a symplectic structure.**

Therefore, shrink $\varepsilon$ if nec. then we have a symplectic mfd

$$\left((-\varepsilon,\varepsilon)\times M,\ \omega|_{TM} + d(r\lambda)\right)$$

cut out from $M \hookrightarrow (X,\omega)$. Moreover, $M$ admits a stable framed Ham str $(\omega,\lambda)$.

//

- Now, let's forget about the ambient symp mfd $(X,\omega)$, and directly consider $(M, (\omega,\lambda))$ where $(\omega,\lambda)$ is stable. Then $\exists\ \varepsilon>0$ s.t.

for this construction "stable" is not needed.

$$\left(\underset{r}{(-\varepsilon,\varepsilon)}\times M,\quad \omega + d(r\lambda)\right)$$

is a symplectic mfd.

<u>Rmk</u>. Recall that if $M$ is a contact mfd, then $\mathbb{R}\times M$ supports a symplectic str called the symplectization of $M$.

Fix a smooth map $\varphi: \mathbb{R}\to(-\varepsilon,\varepsilon)$ s.t. $\varphi'>0$. It induces a map

$$\mathbb{R}\times M \xrightarrow{\Phi_\varphi} (-\varepsilon,\varepsilon)\times M \qquad (r,x)\longrightarrow(\varphi(r),x).$$

Then $\Phi_\varphi^*(\omega+d(r\lambda)) = \omega + d(\varphi(r)\lambda)$.

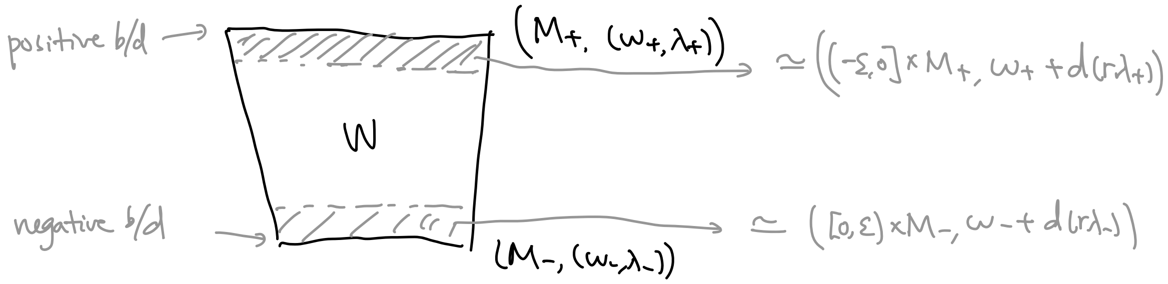
The role of being stable will be explained later.

$\Rightarrow$ <u>Prop</u> Given a $(M,(\omega,\lambda))$ where $(\omega,\lambda)$ is <u>stable</u>, then one can construct a symp mfd (called the <u>symplectization of $(M,(\omega,\lambda))$</u>) by $(\mathbb{R}\times M,\ \omega+d(\varphi(r)\lambda))$

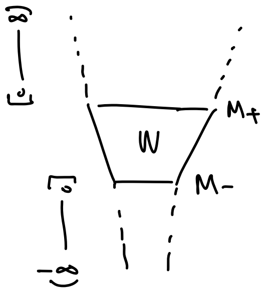
Note that there is <u>no</u> canonical symplectization in this case.

- Similarly in the contact geometry setting, one can generalize symplectization to symplectic cobordisms ( Def ~~6.12~~ Def 6.17 in [Wen]).

<u>Def</u> Given $(M_\pm^{2n-1}, (\omega_\pm, \lambda_\pm))$, a symplectic cobordism with stable boundary from $(M_-, (\omega_-, \lambda_-))$ to $(M_+, (\omega_+, \lambda_+))$ is a cpt symplectic $2n$-dim mfd $W$ s.t. $\partial W \underset{\text{diffeo}}{\simeq} -M_- \amalg M_+$ and $\omega|_{T\partial W} \simeq \omega_\pm$ on $M$ respectively.




Then one can complete $W$:




$\hat{W} := ((-\infty, 0] \times M_-) \cup_{M_-} W \cup_{M_+} ([0, \infty) \times M_+)$ — both can be viewed as part of symplectizations

(where $M_- = \{0\} \times M_-$ and $M_+ = \{0\} \times M_+$)

Equip $\hat{W}$ the following symplectic structure:

$$\omega_\varphi := \begin{cases} \omega_+ + d(\varphi(r)\lambda_+) & \text{on } [0, \infty) \times M_+ \\ \omega & \text{on } W \\ \omega_- + d(\varphi(r)\lambda_-) & \text{on } (-\infty, 0] \times M_- \end{cases}$$

when $\varphi : \mathbb{R} \to (-\varepsilon, \varepsilon)$ s.t. $\varphi' > 0$ and $\underline{\varphi(r) = r}$ near $0$ ← so that one can glue near $M_\pm$ in a symp way.

• Given $(\omega,\lambda)$ where $R_{(\omega,\lambda)}$ is its associated Reeb v.f, consider an a.c.s $J$ on $\mathbb{R}\times M$ satisfying ← there always exists such $J$

(i) $J$ is $\mathbb{R}$-translation invariant.

(ii) $J\partial_r = R_{(\omega,\lambda)}$ and $JR_{(\omega,\lambda)} = -\partial_r$. $\quad (*)$

(iii) $J: \ker\lambda \circlearrowleft$ and $J|_{\ker\lambda}$ is compatible with $\omega|_{\ker\lambda}$.

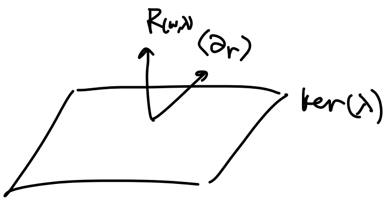
cf. a.c.s on symplectic in contact geo. setting.

We call $J$ <u>is tamed by $(\omega,\lambda)$</u> if $\exists\, \varepsilon>0$ s.t. for every $\varphi: \mathbb{R}\to(-\varepsilon,\varepsilon)$ with $\varphi'>0$, we have $J$ is tamed by $\omega + d(\varphi(r)\lambda)$ on $\mathbb{R}\times M$.

<u>Prop</u> (prop 6.19 in [Wen]) Given a framed Ham str on $M^{2n-1}$, we have

$J$ satisfying $(*)$ is tamed by $(\omega,\lambda)$ $\iff$ $\lambda$ is a stable.

<u>Pf</u>. "$\Longleftarrow$"




$= T(\mathbb{R}\times M)$

and $\omega_\varphi = \omega + d(\varphi(r)\lambda) = \omega + \varphi(r)d\lambda + \varphi'(r)\,dr\wedge\lambda$

Then

- $\omega_\varphi(R_{(\omega,\lambda)}, JR_{(\omega,\lambda)}) = \omega_\varphi(R_{(\omega,\lambda)}, -\partial_r)$

$\overset{(\#)}{=} \varphi'(r)(dr\wedge\lambda)(\partial_r, R_{(\omega,\lambda)}) = \varphi'(r)\lambda(R_{(\omega,\lambda)}) = \varphi'(r) > 0$

(#) is the step using $\lambda$ is stable b/c for the projection $\pi_{\ker\lambda}: \mathbb{R}\times M \to \ker\lambda$ we may have $\pi_{\ker\lambda}(\partial_r) \neq 0$.

The same works for $\partial_r$

d\lambda (v, Jv)

- $\omega_\varphi(v, Jv) = \underbrace{\omega|_{\ker(\lambda)}(v, J|_{\ker(\lambda)}v)}_{\text{by cond of } J \text{ in } (*)\; >0} + \varphi(r)\,dr(v,v) > 0$

$v\in\ker(\lambda)$

if $\varepsilon>0$ and sufficiently small, $\varphi(r)\,dr(v,v)$ is small

"$\Longrightarrow$" Suppose $J$ satisfies (∗) but $\lambda$ is not stable. Then $\exists\, x \in M$ and $v \overset{\neq 0}{\in} \ker(\lambda)(x)$ s.t. $d\lambda(R_{(\omega,\lambda)}, v) > 0$. At $(0,x) \in \mathbb{R}\times M$, for constant $c>0$, we have

$$\omega_\varphi\big(R_{(\omega,\lambda)} + cJv,\ J(R_{(\omega,\lambda)} + cJv)\big) = \omega_\varphi\big(R_{(\omega,\lambda)} + cJv,\ -\partial_r - cv\big)$$

$$= \omega_\varphi(\partial_r, R_{(\omega,\lambda)}) - c\,\omega_\varphi(R_{(\omega,\lambda)}, v) - c\,\omega_\varphi(Jv, \partial_r) + c^2\omega_\varphi(v, Jv)$$

$$= \varphi'(0) - c\big(\omega + \varphi(0)d\lambda + \varphi'(0)(dr\wedge\lambda)\big)(R_{(\omega,\lambda)}, v)$$

$$-c\,(\ \cdot\quad\cdot\quad\cdot\quad\cdot\ )(Jv, \partial_r)$$

$$+c^2(\ \cdot\quad\cdot\quad\cdot\quad\cdot\ )(v, Jv)$$

$$= \varphi'(0) - c\,\varphi(0)\underbrace{d\lambda(R_{(\omega,\lambda)}, v)}_{\substack{\| \\ \gamma}} + c^2\big(\underbrace{\omega(v,Jv)}_{\substack{>0 \\ \| \\ \alpha}} + \varphi(0)\underbrace{d\lambda(v,Jv)}_{\substack{\| \\ \beta}}\big)$$

better to change this \epsilon by \delta, in order not to mess up with the range (-\epsilon, \epsilon) in the range of the reparametrization function \phi.

choose $\varphi(0) = \varepsilon$, then $\quad RHS = \varphi'(0)\ \underbrace{-\,c\cdot\varepsilon\cdot\gamma + c^2(\alpha + \varepsilon\,\beta)}_{(\heartsuit)}$

then choose $c = \varepsilon\cdot\frac{\gamma}{2\alpha}$

$$\Longrightarrow\quad -c\cdot\varepsilon\cdot\gamma + c^2(\alpha + \varepsilon\beta) = -\varepsilon^2\frac{\gamma^2}{2\alpha} + \varepsilon^2\frac{\gamma^2}{4\alpha^2}\cdot\alpha + o(\varepsilon^2)$$

$$= -\varepsilon^2\left(\frac{\gamma^2}{2\alpha} - \frac{\gamma^2}{4\alpha}\right) + o(\varepsilon^2)$$

$$= -\varepsilon^2\cdot\frac{\gamma^2}{4\alpha} + o(\varepsilon^2)$$

So when $\varepsilon$ is sufficiently small, $(\heartsuit) < 0$. Note that $\varepsilon$ can be chosen arbitrarily small. Then pick $\varphi'(0)$ even small, then $\omega_\varphi(R_{(\omega,\lambda)} + cJv, J(R_{(\omega,\lambda)} + cJv)) < 0$ D.

## 4. Energy revisit

$(M^{2n-1}, (\omega, \lambda))$ stable Ham str.

$\mathcal{J}((\omega,\lambda)) = \{$ a.c.s on $\mathbb{R}\times M$ satisfying $(*)\}$

$T_\varepsilon = \{ \varphi : \mathbb{R} \to (-\varepsilon, \varepsilon) \mid \varphi'(0) > 0 \}$ for any fixed $\varepsilon > 0$.
test function

in fact, it should be $\dot{\Sigma} = \Sigma \setminus \{pts\}$

Then for any $J$-hol curve $u : (\Sigma, j) \to (\mathbb{R}\times M, J)$, one can define its energy

$$E_\varepsilon(u) := \sup_{\varphi \in T_\varepsilon} \int_\Sigma u^* \omega_\varphi$$

$\leftarrow$ so $E(u)$ is independent of the test function.

If $J \in \mathcal{J}((\omega,\lambda))$, then $E(u) \geq 0$ and $E(u) = 0$ iff $u$ is constant.

Rmk Def of $E(u)$ is good since "trivial" cylinder admits small energy (cf. Section 1)

Question What's the relation between $E_\varepsilon(u)$ and $E_\delta(u)$ ?

In general, let's consider

$$T_{(a,b)} := \{ \varphi : \mathbb{R} \to (a,b) \subset (-\varepsilon, \varepsilon) \mid \varphi' > 0 \}$$

and $E_{(a,b)}(u) := \sup_{\varphi \in T_{(a,b)}} \int u^* \omega_\varphi$

Prop. $\exists\, C(a,b,\varepsilon)$, independent of $u$, s.t. $C E_\varepsilon(u) \leq E_{(a,b)}(u) \leq E_\varepsilon(u)$.

Pf. The inequality $E_{(a,b)}(u) \leq E_\varepsilon(u)$ is trivial b/c $T_{(a,b)} \subset T_\varepsilon$.