1. Energy

Note that in most studies of J-hol curve $u: (\Sigma, j) \longrightarrow (M, J)$, the target space is an almost cpx mfd.

Question If the target is more specified to be a symplectic mfd (M, ω), what more information can we get ?

This makes sense b/c $\forall$ symp structure ω on M, $\exists$ a contractible set of almost cpx strs $\mathcal{J}(M, \omega)$, so we get (M, ω, J).

Moreover, $\omega(-, J-)$ defines a metric, denoted by g_J.

- By Ex2 in HW1, for any J-hol curve $u: (\Sigma, j) \longrightarrow (M, \omega)$, one can define

$$E(u) := \int_\Sigma u^* \omega.$$

$\Longrightarrow$ when Σ is closed, $E(u) = \langle [\omega], [u(\Sigma)] \rangle$ which means in this case $E(u)$ is a topological quantity.

One uses an ω-compatible a.c.s J to write $E(u)$ more explicitly:

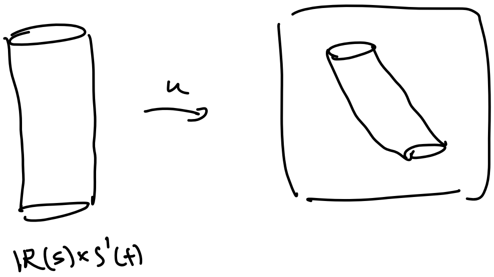
In any local coordinate chart on (Σ, j) choose a basis $\{\partial_s, \partial_t\}$ s.t. $j\partial_s = \partial_t$ and $j\partial_t = -\partial_s$, a metric h on Σ s.t. $\{\partial_s, \partial_t\}$ is an orthonormal basis. Then

$$\begin{aligned} \omega(du(\partial_s), du(\partial_t)) &= \omega((du \cdot j)(\partial_t), du(\partial_t)) \\ &= \omega(-J \cdot du(\partial_t), du(\partial_t)) \quad \text{b/c } u \text{ is J-hol} \\ &= \omega\left(\frac{\partial u}{\partial t}, J\frac{\partial u}{\partial t}\right) = \left|\frac{\partial u}{\partial t}\right|^2_{g_J} \end{aligned}$$

$\Rightarrow$ $E(u)$ is alway non-negative.

(and $E(u) = 0$ iff u is constant in each connected component).

- Another case where energy appears — Ham Floer theory. (M, ω, J, H) with $J = \{J_t\}_{t \in S^1}$, H ← Hamiltonian



u satisfies

$$\frac{\partial u}{\partial s} + J_t(u)\left(\frac{\partial u}{\partial t} - X_H(u(t))\right) = 0 \qquad (*)$$

One often defines the energy of u satisfying $(*)$ by

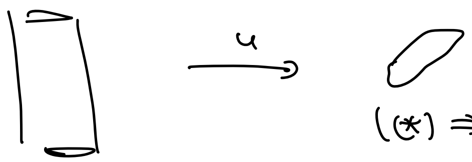
$$E(u) := \frac{1}{2}\int_0^1 \int_{-\infty}^{\infty} \left(\left|\frac{\partial u}{\partial s}\right|_{g_J}^2 + \left|\frac{\partial u}{\partial t} - X_H(u(t))\right|_{g_J}^2 \right) ds\, dt$$

$= \int_{\mathbb{R}\times S^1} u^*\omega$ + extra term given by H

Thm $E(u) < \infty \Leftrightarrow \exists$ closed Hamiltonian orbits $x^{\pm}$ of system (M, ω, J, H) s.t. $\lim_{s \to \pm\infty} u(s,t) = x^{\pm}(t)$.

↑ We will prove this later in this lecture.

Ex. One extreme case of u is "constant loop" i.e. $\frac{\partial u}{\partial s} \equiv 0$,



$x(t) = u(s,t)$ $\forall s$

($(*) \Rightarrow x(t)$ is a closed Ham orbit $\frac{\partial u}{\partial t} = \dot{x}(t) = X_H(x(t))$)

In this case, $E(u) = 0$ (b/c $E(u) = \int_0^1 \int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial s}\right|_{g_J}^2 ds\, dt$).

Ex In general, for $E(u) < \infty$, we have

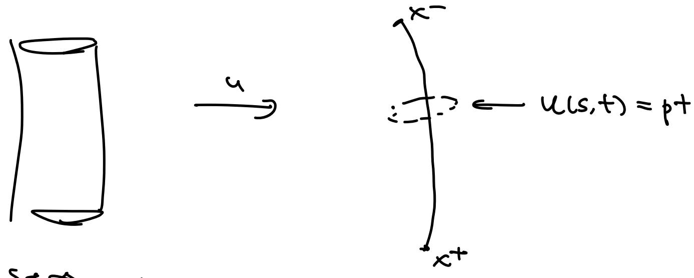
$$\mathcal{A}_H(x^-) - \mathcal{A}_H(x^+) = E(u)$$

← action functional.

↑ this serves as one motivation to define $E(u)$ in the way above.

Ex The other extreme case of u is a "flow line" i.e. $\forall s \in \mathbb{R}$,

$u(s,t) : S^1 \to M$ is constant.

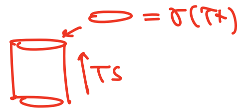


$\Rightarrow$ $u(s,t) \xrightarrow{s \to \pm\infty}$ closed Hamiltonian orbit but as a fixed pt of the Hamiltonian flow $\{\varphi_H^t\}_{t \in [0,1]}$

In this case, $E(u)$ is <u>not</u> nec. zero.

- For symplectization $\left(SX = \mathbb{R}_r \times X,\ d(e^r \alpha)\right)$ where $(X, \xi = \ker\alpha)$ is a contact mfd. We will choose a specific family of a.c.s J.

  Then $u_{trivial} : \mathbb{R} \times S^1 \to SX$ defined by



$$u_{trivial}(s,t) := (Ts, \gamma(Tt))$$

where $\gamma(t)$ is a fixed close Reeb orbit with period T on $(X, \xi = \ker\alpha)$.

- One can check that $u_{trivial}$ is a J-hol curve.
- If one uses the "classical" def of energy

$$\left(\int_\Sigma u^*\omega =\right) \int_{\mathbb{R}\times S^1} u_{trivial}^* d(e^r\alpha) = T^2 \int_{-\infty}^{\infty} e^{Ts}\, ds = +\infty.$$

$\Rightarrow$ This indicates that this is <u>NOT</u> a good definition in this setting.

- Correcting by Hofer: consider functions $\varphi : \mathbb{R} \to (-1,1)$ s.t.

There are many such fcns. $\rightarrow$ $\varphi(0) = 0$, $\varphi' > 0$ and $\varphi(t) \to \pm 1$ as $t \to \pm\infty$.

Then one can check for $d(e^{\varphi(r)}\alpha) =: \omega_\varphi$

$$E_\varphi(u) := \int_{\mathbb{R}\times S^1} u_{\text{trivial}}^* \omega_\varphi = (e-e^{-1})T < \infty.$$

Rmk One can even consider similar fun $\varphi: \mathbb{R} \to (-\varepsilon, \varepsilon)$. So that a "trivial cylinder" has arbitrarily small energy.

2. Stable Hamiltonian structure.

Def M^{2n-1} odd-dim'l mfd. A Hamiltonian str on M is a closed 2-form ω s.t. $\underset{\substack{\cap \\ (2n-2)\text{-form (on } M^{2n-1})}}{\omega^{n-1}} \neq 0$ (i.e. $\forall p \in M$, $\exists$ basis element $v_1, \cdots, v_{2n-2} \in \underset{\dim = 2n-1}{T_pM}$

s.t. $\omega^{n-1}(v_1, \cdots, v_{2n-2}) \neq 0$.)

- Note that by dimension reason, ω is always degenerate in the sense that $\exists$ non-trivial ker, denoted by ξ (which defines a line field on M).

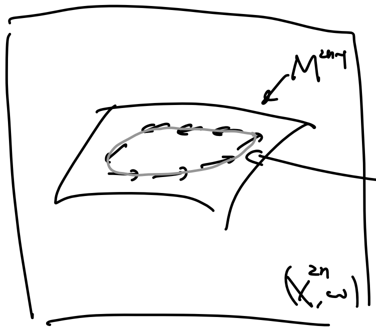
Def (cont) A Hamiltonian str ω on M^{2n-1} is called framed by 1-form $\lambda \in \Omega^1(M)$ if $\lambda(\xi) > 0$. A framed Ham str on M^{2n-1} is a pair (ω, λ).

Note that λ frames ω iff $\lambda \wedge \omega^{n-1} > 0$ (in particular, a volume form on M^{2n-1}) b/c $(\lambda \wedge \omega^{n-1})(\xi, v_{\sigma(1)}, \cdots, v_{\sigma(2n-2)}) = \lambda(\xi)\underbrace{\omega^{n-1}(v_{\sigma(1)}, \cdots, v_{\sigma(2n-2)})}_{>0}$ by reordering the basis elements $\{v_1, \cdots, v_{2n-2}\}$ above.

Rmk (Exe) By POU, for any fixed Ham str ω, framing λ also exists.

__Rmk__ The origin of "Ham" str on M^{2n-1}. For a symp mfd (X^{2n}, ω) and a hypersurface $M^{2n-1} \overset{i}{\hookrightarrow} (X^{2n}, \omega)$, the restriction $i^*\omega$ defines a Ham str b/c M^{2n-1} admits a line field ξ as kernel of $i^*\omega$.

since $i^*\omega$ is degenerate



any integral curve of ξ can be identified as a closed Ham orbit of H s.t. $M = H^{-1}(\{c\})$, a level set.

In $(M^{2n-1}, (\omega, \lambda))$, one can define Reeb v.f. $R_{(\omega,\lambda)}$ by

$$\lambda(R_{(\omega,\lambda)}) = 1 \quad \text{and} \quad \iota_{R_{(\omega,\lambda)}}\omega = 0.$$

$R_{(\omega,\lambda)}$ lies in $\ker \omega$.

$$\Longrightarrow \quad T_pM^{2n-1} \simeq \ker(\lambda_p) \oplus R_{(\omega,\lambda)}(p)$$

and

$$\Longrightarrow \quad \xi = f \cdot R_{(\omega,\lambda)}$$

for some positive fcn $f : M \to \mathbb{R}_{>0}$
(b/c $\lambda(\xi) = f \cdot \lambda(R_{(\omega,\lambda)}) = f > 0$ by def).

__Ex__ $(M^{2n-1}, \ker\alpha)$ contact mfd

Then $(\omega, \lambda) = (d\alpha, \alpha)$ is a framed Ham str on M^{2n-1}.

__Ex__ For $(M^{2n-1}, (\omega, \lambda))$, we have

$$\mathcal{L}_{R_{(\omega,\lambda)}}\omega = d\,\iota_{R_{(\omega,\lambda)}}\omega + \iota_{R_{(\omega,\lambda)}}d\omega = 0$$

So $\varphi^t_{R(\omega,\lambda)}$ preserves ω.

Studies on closed Reeb orbits (or Reeb flow) (in $(M, (\omega,\lambda))$-setting) is similar to those in contact mfd.

$\Rightarrow$ One can ask the same question about the existence of <u>closed</u> Reeb orbits of a framed Ham str. (cf. Weinstein conjecture).

Thm (Hutchings-Taubes, 09) For $(M^3_{\text{closed}}, (\omega,\lambda))$, if (ω,λ) is <u>stable</u> (later) then if M is <u>not</u> a $\mathbb{T}^2$-bundle over S^1, then $\exists$ at least one closed Reeb orbit.

<u>Ex</u> $(X^2, \tilde{\omega})$ closed symplectic mfd. Consider trivial fiber bundle $M = X \times S^1 \xrightarrow{\pi} S^1$

For $d\theta$ (closed 1-form on S^1), define

$$\lambda = \pi^* d\theta \quad \text{and} \quad \omega := \pi_X^* \tilde{\omega} \quad \text{where } \pi_X : X \times S^1 \to X$$

Then ω is closed and $\omega \neq 0$ (b/c one can choose basis $\{e_1, e_2\}$ on X-part).

Moreover, $\lambda \wedge \omega = \left(\pi^* d\theta \wedge \pi_X^* \tilde{\omega}\right)\left(\frac{\partial}{\partial\theta}, e_1, e_2\right)$

$$= d\theta\left(\frac{\partial}{\partial\theta}\right) \tilde{\omega}(e_1, e_2) > 0.$$

Note that $R_{(\omega,\lambda)} = \{0\} \times \frac{\partial}{\partial\theta}$:

$$\lambda\left(R_{(\omega,\lambda)}\right) = \left(\pi^* d\theta\right)\left(\{0\} \times \frac{\partial}{\partial\theta}\right) = d\theta\left(\frac{\partial}{\partial\theta}\right) = 1$$

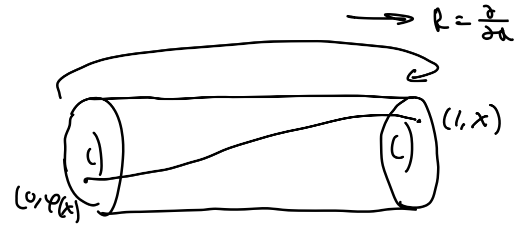
$$\omega\left(R_{(\omega,\lambda)}, -\right) = \left(\pi_X^* \tilde{\omega}\right)\left(\{0\} \times \frac{\partial}{\partial\theta}, -\right) = 0.$$

This example extends in this way: consider a symplectomorphism $\varphi : (X^2, \omega) \circlearrowleft$, then

$$Y_\varphi := \frac{X^2 \times [0,1]}{(1,x) \sim (0, \varphi(x))}$$

(called mapping torus of φ on (X^2, ω))

Pictorially,



$\xrightarrow{\Pi_X} (X^2, \omega)$

$\downarrow \pi$

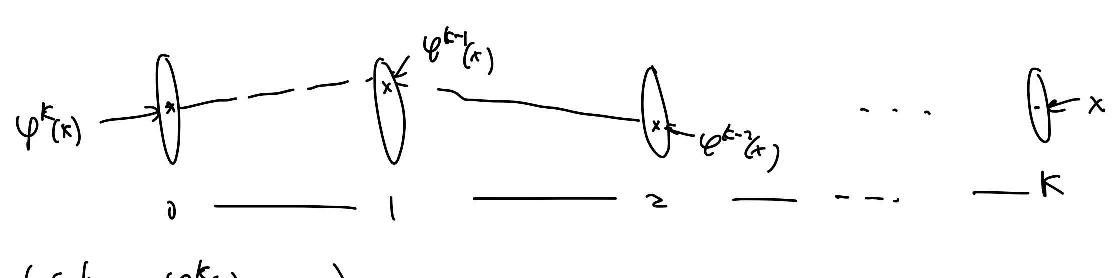
$(S^1, d\theta)$

Then again, define $\lambda := \pi^* d\theta$ and $\omega_\varphi := \Pi_X^* \omega$, then $(\omega_\varphi, \lambda)$ is a framed Hamiltonian str on Y_φ.

Also, one can check that the Reeb v.f of $(\omega_\varphi, \lambda)$ is $R = \frac{\partial}{\partial\theta}$.

$\Longrightarrow$ $\left\{ \begin{array}{c} \text{closed Reeb orbits} \\ \text{of } (\omega_\varphi, \lambda) \text{ on } Y_\varphi \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{fixed pts of } \varphi \\ \text{on } (X, \omega) \end{array} \right\}$

Rmk "fixed pts" better to be "periodic pts" (of period k)



(where $\varphi^k(x) = x$)

Rmk Learned from Guanheng Chen's arXiv:2111.11891: If φ is generated