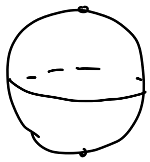
e.g.



$S^2 = \overline{D} \cup \underline{D}$

$\overline{D} = S^2 \setminus \{\text{south pole}\}$, $\quad \underline{D} = S^2 \setminus \{\text{north pole}\}$

$\overline{D} \xrightarrow{\overline{\Phi}} \overline{\mathbb{C}} \times \mathbb{C}$

$\underline{D} \xrightarrow{\underline{\Phi}} \underline{\mathbb{C}} \times \mathbb{C}$

+ glue via transition map φ_k

For $(z,v) \in \overline{\mathbb{C}} \cap \underline{\mathbb{C}}$, $\underbrace{\underline{\Phi} \circ \overline{\Phi}^{-1}}_{\text{transition map}} \Big|_{\overline{\mathbb{C}} \cap \underline{\mathbb{C}}} (z,v) := \left(\frac{1}{z}, \frac{v}{z^k}\right)$ ← this is holomorphic.

$\overline{\mathbb{C}} \cap \underline{\mathbb{C}} = \mathbb{C} \setminus \{0\}$

Then this transition map defines a cpx vector bundle $\begin{matrix} E_k \\ \downarrow \\ S^2 \end{matrix}$ and $c_1(E_k) = k$. (Exe)

- What's special of our D_u ?

$$D_u \xi = \frac{1}{2}\Big(\underbrace{\nabla \xi + J(u) \nabla \xi \cdot j}_{\substack{\text{locally, } \frac{1}{2}(d\xi + J \cdot d\xi \cdot j) = \bar{\partial}\xi \\ \text{standard cpx Cauchy-Rieman operator}}} + \underbrace{(\nabla_\xi J) \cdot Tu \cdot j}_{\text{some operator } A \in \Omega^{0,1}(\Sigma, \mathrm{End}_{\underline{\underline{\mathbb{R}}}}(u^*TM))}\Big)$$

this is the part that prevents D_u to be always cpx-linear.

Since Σ is closed, one can check that such A is a cpt operator.

$\Longrightarrow$ $\mathrm{ind}(D_u) = \mathrm{ind}(\bar{\partial})$ the Fredholm index of a cpx Cauchy-Riem operator!

For a complex vector bundle $\begin{matrix} (E,J) \\ \downarrow \\ (\Sigma,j) \end{matrix}$ and a $\underline{\text{cpx}}$ Cauchy-Riem operator D,

by def. $\mathrm{Ker}(D) := \{$ holomorphic sections of this bundle $\}$.

and $\mathrm{coker}(D) = \mathrm{Ker}(D^*)$ where $D^* : \Omega^{0,1}(\Sigma, E) \longrightarrow \Omega^0(\Sigma, E)$.

$\underline{\text{FACT}}$ (proof is based on Hodge theory) D^* is conjugate to a cpx Cauchy-Riem operator on $\left((T^*\Sigma)^{0,1}_{\mathbb{C}} \otimes_{\mathbb{C}} E, \underline{\underline{-J}}\right) =: (\hat{E}, \hat{J})$

$\quad\downarrow$

(Σ, j)

negate a.c.s.

Note that $(T^*\Sigma)^{0,1}_{\mathbb{C}} \otimes ((T^*\Sigma)^{1,0}_{\mathbb{C}} \otimes E) \simeq E$

b/c $(T^*\Sigma)^{1,1}_{\mathbb{C}}$ is trivial $(\simeq \mathbb{C})$

Rmk when Σ is non-cpt, then A above may not be cpt.

Rmk $$c_1(\hat{E}) = \underset{\substack{\uparrow \\ \text{from } -J}}{-} c_1\big((T^*\Sigma)^{0,1}_{\mathbb{C}} \otimes_{\mathbb{C}} E\big) = -\text{rank}_{\mathbb{C}}(E)\, c_1\big(\overset{\text{this is a line bundle}}{(T^*\Sigma)^{0,1}_{\mathbb{C}}}\big) - c_1(E)$$
$$= -\text{rank}_{\mathbb{C}}(E)\ \chi(\Sigma) - c_1(E)$$

← 负负得正 ① $c_1(T^*\Sigma) = -c_1(T\Sigma)$ ② $(T^*\Sigma)^{0,1}_{\mathbb{C}}$ anti-hol part

The discussion above says that if D is Fredholm, then ind(D) is completely determined by (i) top type of Σ and (ii) $c_1(E)$.

Thm Given a cpx Cauchy-Rieman operator D on $\begin{array}{c}(E,J)\\ \downarrow\\ (\Sigma,\underset{\text{closed}}{j})\end{array}$ where $n = \text{rank}_{\mathbb{C}} E$, if D is Fredholm, then $\underset{\substack{\uparrow\\ \text{real dim}}}{\text{ind}(D)} = n\chi(\Sigma) + 2c_1(E)$.

(This thm is one version of Rieman-Roch Thm).

Rmk Observe that ind(D) is alway even!

4. Proof of index formula

* There are two approaches to present this:

- Identify the index formula with the classical expression of Rieman-Roch and then directly apply. Then one needs to introduce quite a lot of notations (which is based on divisors). In particular, one needs to identify line bundles with divisors.

we will take this approach.

- Take a non-rigorous but with relatively more detailed proof from Wendl's another book (note - Lectures on Holomorphic curves) - Thm 3.22.

Lemma Suppose $\begin{array}{c}(E,J)\\ \downarrow\\ (\Sigma,j)\end{array}$ is a cpx line bundle, then

(1) if $c_1(E) < 0$, then D is injective

(2) if $c_1(E) > -\chi(\Sigma)$, then D is surjective

Pf. (1) Suppose $\exists\, \xi (\neq 0) \in \ker D$, then ξ, as a $\overset{\text{or (J-hol)}}{\wedge}$ holomorphic map must has isolated zeros. Moreover, each zero is counted in a positive way (positivity intersection). $\Longrightarrow$ $c_1(E) \geq 0$.

(2) Consider D^* (on bundle $\begin{array}{c}(\hat{E},\hat{J})\\ \downarrow \\ (\Sigma,j)\end{array}$), then $c_1(\hat{E}) < 0 \Rightarrow D^*$ inj $\Longleftrightarrow D$ surj.

Here $c_1(\hat{E}) = -\chi(\Sigma) - c_1(E) < 0 \Longleftrightarrow c_1(E) > -\chi(\Sigma)$. $\square$

The proof of the index formula splits into a few steps.

① $n=1$ line bundle and $\Sigma = S^2$.

- By lemma above, $\chi(\Sigma) = 2$, so either $c_1(E) < 0$ (≤ -1) or $c_1(E) > -2$ (≥ -1), so either D is injective or surjective

  In particular, if $c_1(E) = -1$, then D is an iso.

  By switching to D^*, we can focus on the case where D is surjective.

  so $\operatorname{ind}(D) = \dim \ker(D)$

since we will study hol section, only need to consider $k \geq 0$.

- Example above constructs a cpx line bundle $\begin{array}{c}(E_k, J)\\ \downarrow \\ (S^2, j)\end{array}$ s.t. $c_1(E_k) = k$, so any other cpx line bundle with the same c_1 must be iso to E_k

  Then for E_k, one can compute by hand,

$$\ker(D) = \{\text{holomorphic sections}\}$$

has $\dim_{\mathbb{R}} = 2 + 2k$. $\Longrightarrow$ $\operatorname{ind}(D) = \dim \ker(D) = 2+2k = 1 \cdot \chi(S^2) + 2c_1(E_k)$

$\uparrow$ b/c any hol section $= \sum_{i=0}^{k} a_i z^i$ $a_i \in \mathbb{C}$

② For general $n \geq 2$, consider rank n cpx vector bundle

$$E := \underbrace{\overbrace{E_{-1} \oplus \cdots \oplus E_{-1}}^{n-1} \oplus E_k}_{n\text{-piece}}$$

- Then $c_1(E) = k-(n-1)$. Any cpx Cauchy-Riemann operator $D=(D_1,\dots,D_{n-1},D_n)$ where $D_1,\dots,D_{n-1}$ are all isomorphisms. For D_n, by Lemma above again, $\underset{\text{(so is } D)}{D_n}$ is injective if $k \leq -1$ and $\underset{\text{(so is } D)}{D_n}$ is surjective if $k \geq -1$.

- Up to duality, assume D is surjective (and $k \geq 0$). Then

$$\begin{aligned} \operatorname{ind}(D) = \dim \operatorname{Ker}(D) &\overset{\dim}{=} \left(\{\text{hol } \underset{\bar{\partial}}{\text{sections}} \text{ of } E_1\} \oplus \cdots \oplus \{\text{hol setions of } E_k\}\right) \\ &= 0 + \cdots + 0 + 2 + 2k \\ &= n \cdot 2 + 2(k-(n-1)) \\ &= n \cdot \chi(S^2) + 2c_1(E). \end{aligned}$$

③ Rmk. For general Σ_g, it's not always true that D is either inj or surjective. Also, computing the hol sections is not that easy. Modulo these difficulties, let's assume we know $\operatorname{ind}(D)$ is in the form

↙ only depending on Σ and rank(E), independent of D.

$$\operatorname{ind}(D) = C(\Sigma, n) + 2c_1(E) \quad \text{for some constant } C(\Sigma,n).$$

- Then

$$\begin{aligned} \operatorname{ind}(D^*) &= C(\Sigma,n) + 2c_1(\hat{E}) \\ &= C(\Sigma,n) - 2n\chi(\Sigma) - 2c_1(E) \end{aligned}$$

So, $\operatorname{ind}(D) = -\operatorname{ind}(D^*) \Rightarrow C(\Sigma,n) + 2c_1(E) = -C(\Sigma,n) + 2n\chi(\Sigma) + 2c_1(E)$

$$\Rightarrow C(\Sigma,n) = n\chi(\Sigma).$$

- Finally, $\operatorname{ind}(D) = n\chi(\Sigma) + 2c_1(E)$. ☐

<u>Ex</u> Back to the concrete case when $D = \overset{u:(\Sigma,j)\to(M^{2n},J)}{D_{u,A}}$ when there is a top constrain that $[u(\Sigma)] = A \in H_2(M;\mathbb{R})$. Then for generic J,

$$\underset{\text{manifold}}{\overset{\uparrow}{\dim}}\, M_{J,A} = \operatorname{ind}(D_{u,A}) = n\cdot\chi(\Sigma) + 2c_1(u^*TM)$$

In general, one defines the "virtual" dim of $M_{J,A}$ by $\operatorname{ind}(D_{u,A})$.

Recall the naturality of the chern class:

$$\begin{tikzcd} u^*TM \arrow[r, dashed] \arrow[d, dashed] & TM \arrow[d] \\ \Sigma \arrow[r, "u"] & M \end{tikzcd} \implies c_1(u^*TM) = u^*c_1(TM)$$

where $c_1(TM) \in H^2(M;\mathbb{Z})$.

Then $\big(u^*c_1(TM)\big)([\Sigma]) = c_1(TM)\big([u(\Sigma)]\big) = c_1(TM)(A)$.

Hence, $\dim M_{J,A} = n\cdot\chi(\Sigma) + 2\underbrace{c_1(TM)(A)}_{\text{usually denoted by } c_1(A)}$.

Eventually, we remark that the "brief argument" for generic $J \Rightarrow M_{J,A}$ is a mfd only works for simple curves.

<u>Thm</u> Given (Σ, j), for a generic J on M^{2n} and fixed $A \in H_2(M;\mathbb{Z})$,

$$M^*_{J,A} = \{\text{simple } J\text{-hol } u:(\Sigma,j)\to(M,J),\ [u]=A\}$$

is a mfd with $\dim_{\mathbb{R}} = n\chi(\Sigma) + 2c_1(A)$.

Here is a simple reason why this statement can <u>not</u> include multiple covers:

<u>Ex</u>.

$$\begin{tikzcd} \mathbb{T}^2 = \tilde{\Sigma} \arrow[r, "u"] \arrow[d, "\varphi\ \deg(\varphi)=2"] & (M,J) \\ S^2 = \Sigma \arrow[ur, "v"'] & \end{tikzcd}$$

←

Note that this does <u>not</u> contradict the Riemann-Hurwitz formula.

$$2g(\tilde{\Sigma}) - 2 = \deg(\varphi)\,(2g(\Sigma)-2) + \text{ramification}$$

$$(2\cdot 1 - 2 = 2(2\cdot 0 - 2) + 4\ \nearrow)$$

Assume v is simple and J-hol, representing class A.

Then if $u = v \circ \varphi$ is also J-hol, then it representing class $2A$.

$$\Longrightarrow \quad \dim_{\mathbb{R}} \mathcal{M}_{J,2A} \geq \dim_{\mathbb{R}} \mathcal{M}_{J,A}$$ (if thm above works for moduli space with all curves, including multiple covers)

$$\Longleftrightarrow \quad n(2-2\cdot 1) + 2c_1(2A) \geq n(2-2\cdot 0) + 2c_1(A)$$

$$\Longleftrightarrow \quad 2c_1(A) \geq 2n$$

$$\Longleftrightarrow \quad c_1(A) \geq n$$

But a priori, we don't put any constraint of A (and very likely $c_1(A)$ could be very small).

<u>Rmk</u> This example also indicates that the bigger $c_1(A)$ is, the less trouble the multiple covers will cause. (cf. the third condition in the def of semi-positivity of a symp mfd (Hofer-Salamon)).

<u>Cor</u> Given a generic path $\{J_t\}_{t\in[0,1]}$ (connecting J_0 and J_1), and class $A \in H_2(M,\mathbb{Z})$,

$$\dim_{\mathbb{R}} \mathcal{M}^*_{\{J_t\},A} = \dim_{\mathbb{R}} \left\{ (t,u) \;\middle|\; \bar\partial_{J_t} u = 0,\ \text{simple},\ [u]=A \right\}$$
$$= 1 + 2\chi(\Sigma) + 2c_1(A)$$

Here, generic means, $(t,u) \to \bar\partial_{J_t} u$ has surjective linearization at any J_t-hol u.

<u>Rmk</u> Often one encounters the following notation (see

$$\mathcal{M}^*_{g,\underline{\underline{k}}}(A;J) = \frac{\mathcal{M}^*_{A,J} \times \left((\Sigma_g)^k \setminus \Delta\right)}{G}$$

diagonal

$\leftarrow$ extra decoration

$G \leftarrow$ automorphism group of Σ_g acting diagonally

This notation includes marked pts, used for GW invariant.