Thm Given $u : (\overset{\swarrow \text{closed}}{\Sigma}, j) \longrightarrow (M, J)$ J-hol, there exists $(\overset{\swarrow \text{closed}}{\Sigma'}, j')$ and a J-hol map $v : (\Sigma', j') \to (M, J)$ and a holomorphic map $\varphi : (\Sigma, j) \to (\Sigma', j')$ s.t.

$$\begin{tikzcd}
(\Sigma, j) \arrow[r, "u"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & (M, J) \\
(\Sigma', j') \arrow[ur, "v"'] &
\end{tikzcd}$$

$\deg(\varphi) \geq 1 \longrightarrow \varphi$

where v is an embedding except for possibly finitely many pts (from self-intersection pts and critical pts).

$\Delta(v)$ $\quad$ $\mathrm{Crit}(v)$

Sketch of proof. Consider $\dot{\Sigma}' = u\left(\Sigma \setminus \left(\mathrm{Crit}(u) \cup \Delta(u)\right)\right)$ ← finitely many

Then

$$(\dot{\Sigma}', j') \xrightarrow{\text{inclusion}} (M, J)$$

$\swarrow$

$$\Sigma' := \dot{\Sigma}' \cup_{\Sigma} \coprod_{\mathrm{Crit}(u) \cup \Delta(u)/\sim} D$$

where D are (disjoint) NBH of (equ class of) pts in $\mathrm{Crit}(u) \cup \Delta(u)$.

Here $\sim$ means under map u, NBHs coincide under u.

Then $(\Sigma', j') \longrightarrow (M, J)$ J-hol except finitely many "singular pts".

Finally. $u|_{\Sigma \setminus (\mathrm{Crit}(u) \cup \Delta(u))} \xrightarrow{\text{hol}} \dot{\Sigma}'$ $\rightsquigarrow$ extend by removal of singularities.

$$\begin{tikzcd}
(\Sigma, j) \arrow[r, "u"] \arrow[d, "\varphi"'] & (M, J) \\
(\Sigma', j') \arrow[ur, "v"'] &
\end{tikzcd}$$

u + removal of sing. $= \varphi$

almost like an inclusion

$u(\Sigma \setminus \{\text{sing}\}) = (\Sigma', j')$

$\uplus$ image of NBH of sing.

$\square$

Thm $\Rightarrow$ if $\deg(\varphi)=1$, then φ is a diffeomorphism (invertible in particular),

so on most part of Σ, u is ~~like an inclusion, so~~ injective

$\Rightarrow \exists\, z\in\Sigma$ s.t. $u^{-1}(\{u(z)\}) = z$ and $z \notin \mathrm{Crit}(u)$. $(***)$

<u>Def</u> For $u:(\Sigma, j)\to(M, J)$, any $z\in\Sigma$ satisfying $(***)$ is called an injective pt. Moreover, u is called <u>somewhere injective</u> if $\exists$ at least one injective pt.

not injective $\Leftrightarrow$ either $\# u^{-1}(\{u(z)\}) \geq 2$ or $z \in \mathrm{Crit}(u)$

Observe that if u is somewhere injective, then $\deg(\varphi)=1$

For case $\deg=1$, we call $u:(\Sigma, j)\to(M,J)$ <u>simple</u> (then φ in Thm above is just a reparametrization). Otherwise, u is called a <u>multiple cover</u> — the # of fold is determined by either $\# u^{-1}(\{u(z)\})$ or the local degree of the power when $z\in \mathrm{Crit}(u)$

Discussion above says: $\boxed{\underbrace{\text{somewhere injective}}_{\substack{\text{local}\\ \text{(singularity)}}} \Leftrightarrow \underbrace{\text{simple}}_{\substack{\text{global}\\ \text{(top)}}}.}$

<u>Rmk</u> When studying moduli space, multiple covers usually brings a lot of trouble. So often one only focus on moduli space of simple curves. Verifying u being simple is a local checking.

<u>Rmk</u> Thm above also says for any J-holo curve, $\exists$ an underlying $\underline{\underline{\text{simple}}}$ J-hol curve.

3. General Cauchy-Rieman operator

← revisiting the end of SFT-2

Recall in complex analysis, the Cauchy-Rieman eqn for $\xi : \underset{\underset{\mathbb{R}^2(s,t)}{\cong}}{\mathbb{C}} \to \mathbb{C}$:

$$\frac{\partial \xi}{\partial s} + \sqrt{-1}\frac{\partial \xi}{\partial t} = 0 \quad \left(\Leftrightarrow \frac{\partial \xi}{\partial \bar{z}} = 0\right)$$

One can view this from operator perspective (note that $\frac{\partial \xi}{\partial s} = d\xi(\partial_s)$):

$$\xi \in \Omega^0(\mathbb{C}) \xrightarrow[\text{derivative}]{\text{exterior}} d\xi \in \underset{\underset{\Omega^{(1,0)}(\mathbb{C}) \oplus \Omega^{(0,1)}(\mathbb{C})}{\shortparallel}}{\Omega^1(\mathbb{C})} \xrightarrow{\text{projection}} \underbrace{\frac{1}{2}(d\xi + \sqrt{-1}\cdot d\xi \cdot \sqrt{-1})}_{=:\, \bar{\partial}\xi \,\leftarrow\, \bar{\partial}_{\sqrt{-1}}\xi.} \in \Omega^{0,1}(\mathbb{C})$$

Then

$$\frac{1}{2}(d\xi + \sqrt{-1}\cdot d\xi\cdot\sqrt{-1})(\partial_s) = \frac{1}{2}\left(\frac{\partial \xi}{\partial s} + \sqrt{-1}\frac{\partial \xi}{\partial t}\right) = 0$$

$$\Updownarrow$$

$$\frac{1}{2}(d\xi + \sqrt{-1}\cdot d\xi\cdot\sqrt{-1})(\partial_t) = \frac{1}{2}\left(\frac{\partial \xi}{\partial t} - \sqrt{-1}\frac{\partial \xi}{\partial s}\right) = 0$$

Therefore $\bar{\partial} : \Omega^0(\mathbb{C}) \longrightarrow \Omega^{0,1}(\mathbb{C})$

<u>Rmk</u> $\Omega^{0,1}(\mathbb{C}) = \Gamma(\mathbb{C}, \sqrt{-1}\text{-eigenspace of } T^*\mathbb{C} \circlearrowleft \sqrt{-1}.)$

Note that $\bar{\partial}$ satisfies: $\forall \xi \in \Omega^0(\mathbb{C})$, $f \in \Omega^0(\mathbb{C})$, we have

$$\begin{aligned}
\bar{\partial}(f\xi) &= \frac{1}{2}(d(f\xi) + \sqrt{-1}\cdot d(f\xi)\cdot\sqrt{-1}) \\
&= \frac{1}{2}(f d\xi + \xi\cdot df + \sqrt{-1}\cdot(f d\xi + \xi\, df)\cdot\sqrt{-1}) \\
&= f\cdot\frac{1}{2}(d\xi + \sqrt{-1}\cdot d\xi\cdot\sqrt{-1}) + \frac{1}{2}(df + \sqrt{-1}\cdot df\,\sqrt{-1})\cdot\xi \\
&= f\cdot\bar{\partial}\xi + \bar{\partial}f\cdot\xi
\end{aligned}$$

Now, we generalize this to more general setting:

$\xi \in \Omega^0(\mathbb{C})$ $\shortparallel$ $\mathbb{C}$-valued function

$$\begin{array}{c} \mathbb{C}\times\mathbb{C} \leftarrow \text{bundle} \\ \downarrow \circlearrowright \xi = \text{section} \\ \mathbb{C} \end{array} \Bigg) \text{generalize} \quad \begin{array}{ccc} \mathbb{C}\times\mathbb{C} & \rightsquigarrow & E \\ \downarrow & & \downarrow \\ \mathbb{C} & \rightsquigarrow & \Sigma \end{array}$$

Consider a complex bundle $\begin{matrix}(E, J)\\ \downarrow\\ (\Sigma, j)\end{matrix}$ ← recall that a complex bundle means each fiber (vector space) admits a complex str $J^2 = -\mathbb{1}$.

For this lecture, assume (Σ, j) is a closed Riem surface, j is an a.c.s.

$\Longrightarrow \quad \Omega^0(\Sigma), \ \Omega^1(\Sigma), \ \dots$

$$\Gamma(\Sigma, E) = \Omega^0(\Sigma, E) = \left\{ \text{sections of } \begin{matrix} E \\ \downarrow \\ \Sigma \end{matrix} \right\}$$

$$\Omega^1(\Sigma, E) = \left\{ \text{section of } \begin{matrix} \underbrace{(T^*\Sigma)_{\mathbb{C}}}_{\text{complexification}} \otimes_{\mathbb{C}} E \\ \downarrow \\ \Sigma \end{matrix} \right\} = \Omega^{1,0}(\Sigma, E) \oplus \Omega^{0,1}(\Sigma, E)$$

Then a cpx-linear Cauchy-Rieman operator on $\begin{matrix}(E, J)\\ \downarrow\\ (\Sigma, j)\end{matrix}$ is a $\mathbb{C}$-linear operator

$$D : \Omega^0(\Sigma, E) \longrightarrow \Omega^{0,1}(\Sigma, E)$$ ← module over $\Omega^0(\Sigma)$

satisfying $D(f\xi) = f(D\xi) + \underbrace{(\bar\partial f)}_{df + \sqrt{-1}\cdot df\cdot j} \otimes \xi$ for any $\xi \in \Omega^0(\Sigma, E)$ and $f \in \Omega^0(\Sigma)$.

Rmk A real-linear Cauchy-Rieman operator is defined in the same way but the Lebniz rule holds only for $f : \Sigma \to \mathbb{R}$. ← Then $\bar\partial f$ is still defined by $df + \sqrt{-1}\cdot df\cdot j$ mapped into $\Gamma(\Sigma, \underbrace{(T^*\Sigma)_{\mathbb{C}}}_{\text{complexification}})$

Ex By SFT-2, if $u : (\Sigma, j) \to (M, J)$ J-hol, then the linearization

D_u (of $\bar\partial_J u$) : $\Omega^0(\Sigma, u^*TM) \longrightarrow \Omega^{0,1}(\Sigma, u^*TM)$ is a real-linear Cauchy-Rieman operator on the complex bundle $\begin{matrix}(u^*TM, u^*J)\\ \downarrow\\ (\Sigma, j)\end{matrix}$.

Rmk

$$\left\{ \begin{matrix} \text{cpx-linear Cauchy-Riemann} \\ \text{operators on } \begin{matrix}(E, J)\\ \downarrow\\ (\Sigma, j)\end{matrix} \end{matrix} \right\} \xleftrightarrow[\text{correspondence}]{\text{bijective}} \left\{ \begin{matrix} \text{holomorphic structure} \\ \text{on bundle } \begin{matrix}(E, J)\\ \downarrow\\ (\Sigma, j)\end{matrix} \end{matrix} \right\}$$

i.e. transition map $\underset{(\Sigma, j)}{U \cap V} \to GL(n, \mathbb{C})$ is holomorphic.

– How to describe the complexity of complex vector bundles?

Ref (Thm 2.69 in McDuff–Salamon's red book, 1998)

There exists a functor c_1 (called the first Chern number), unique, from cat of cpx vector bundles over closed (Σ, j) to $\mathbb{Z}$, satisfying the following axioms:

① $$\begin{array}{c} (E,J) \simeq (E',J') \\ \searrow \quad \swarrow \\ (\Sigma, j) \end{array}$$ iff $c_1(E) = c_1(E')$ and $rk(E) = rk(E')$

② $$\begin{array}{ccc} (\varphi^* E, \varphi^* J) & \to & (E,J) \\ \downarrow & & \downarrow \\ (\Sigma', j') & \xrightarrow{\varphi} & (\Sigma, j) \end{array}$$ $\Rightarrow$ $c_1(\varphi^* E) = \deg(\varphi) \cdot c_1(E)$

③ $$\begin{array}{cc} (E_1, J_1) & (E_2, J_2) \\ \downarrow & \downarrow \\ (\Sigma, j) & (\Sigma, j) \end{array}$$ $\Rightarrow$ $c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2)$

Note that $c_1 \equiv 0$ satisfies ~~these~~ these three axioms.

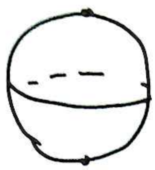
④ (normalization) $$\begin{array}{c} (T\Sigma, J = j_*) \\ \downarrow \\ (\Sigma, j) \end{array}$$ $\Rightarrow$ $c_1(T\Sigma) = 2 - 2g(\Sigma)$.

Rmk One can deduce that $c_1(E) = 0$ iff $\begin{array}{c}(E,J)\\ \downarrow \\ (\Sigma, j)\end{array}$ is a trivial bundle

Also, $c_1(E \otimes E') = c_1(E)\,\mathrm{rank}(E') + c_1(E')\,\mathrm{rank}(E)$.

For $\mathrm{rank}(E) = 1$, a complex line bundle, a more geometric interpretation of $c_1(E)$ is the $\#\{\text{zero of a generic section}\}$.

e.g. $$\begin{array}{c} E = S^2 \times \mathbb{C} \\ \downarrow \quad \uparrow \\ \Sigma = S^2 \end{array}$$ any generic section admits NO zero $\Rightarrow$ $c_1(S^2 \times \mathbb{C}) = 0$.

e.g. 

$$S^2 = \underset{S^2\setminus\{\text{south pole}\}}{\bar{D}} \cup \overset{S^2\setminus\{\text{north pole}\}}{D}$$

$$\bar{D} \xrightarrow{\bar{\Phi}} \bar{\mathbb{C}} \times \mathbb{C}, \qquad D \xrightarrow{\Phi} \mathbb{C} \times \mathbb{C}$$

+ glue via transition map φ_K

For $(z,v) \in \underset{\mathbb{C}\setminus\{0\}}{\bar{\mathbb{C}} \cap \mathbb{C}}$, $\underbrace{\bar{\Phi} \circ \Phi^{-1}\big|_{\bar{\mathbb{C}}\cap\mathbb{C}}}_{\text{transition map}} (z,v) := \left(\frac{1}{z}, \frac{v}{z^k}\right)$ ← This is holomorphic.

Then this transition map defines a cpx vector bundle $\begin{matrix} E_k \\ \downarrow \\ S^2 \end{matrix}$ and $c_1(E_k) = k$. (Exe)

- What's special of our D_u?

$$D_u \xi = \frac{1}{2}\left(\underbrace{\nabla \xi + J(u)\nabla\xi \cdot j}_{} + \underbrace{(\nabla_\xi J)\cdot Tu \cdot j}_{}\right)$$

This is the part that prevents D_u to be always cpx-linear.

locally, $\frac{1}{2}(d\xi + J \cdot d\xi \cdot j) = \bar{\partial}\xi$ — standard cpx Cauchy-Riemann operator (induced by ∇)

some operator $A \in \Omega^{0,1}(\Sigma, \mathrm{End}_{\underline{\underline{\mathbb{R}}}}(u^*TM))$

Since Σ is closed, one can check that such A is a cpt operator.

$\Longrightarrow$ $\mathrm{ind}(D_u) = \mathrm{ind}(\bar{\partial})$ the Fredholm index of a cpx Cauchy-Riem operator!

For a complex vector bundle $\begin{matrix}(E,J)\\ \downarrow \\ (\Sigma,j)\end{matrix}$ and a $\underline{\text{cpx}}$ Cauchy-Riem operator D,

by def. $\mathrm{Ker}(D) := \{$holomorphic sections of this bundle$\}$.

and $\mathrm{coker}(D) = \mathrm{Ker}(D^*)$ where $D^*: \Omega^{0,1}(\Sigma, E) \to \Omega^0(\Sigma, E)$.

<u>FACT</u> (proof is based on Hodge theory) D^* is conjugate to a cpx Cauchy-Riem operator on $\begin{matrix}((T^*\Sigma)^{0,1}_{\mathbb{C}} \otimes_{\mathbb{C}} E, -J) =: (\hat{E}, \hat{J}) \\ \downarrow \\ (\Sigma, j)\end{matrix}$

Note that $(T^*\Sigma)^{0,1}_{\mathbb{C}} \otimes ((T^*\Sigma)^{0,1}_{\mathbb{C}} \otimes E) \simeq E$ b/c $(T^*\Sigma)^{0,2}_{\mathbb{C}}$ is trivial.