Here is a useful corollary of Carleman similarity principle.

Recall any holomorphic map $u: D \to \mathbb{C}^n$ admit local Taylor expansion near pt $0 \in D$.

$$u(z) = a_0 + a_1 z + \frac{a_2^2}{2!} z^2 + \cdots \qquad \text{(only involving power of } z\text{)}$$

Then if $\lim_{|z| \to 0} \frac{|u(z)|}{|z|^k} = 0$ for every $k \in \mathbb{N}_{\geq 0}$ then $u \equiv 0$ near 0.

$\Rightarrow$ for two holomorphic fcns $u_0, u_1 : D \to \mathbb{C}^n$,

$$\lim_{|z| \to 0} \frac{|u_0(z) - u_1(z)|}{|z - z_0|^k} = 0 \text{ for every } k \in \mathbb{N}_{\geq 0} \Rightarrow u_0 = u_1 \text{ near } z_0.$$

at z_0, u_0 and u_1 "agree to infinite order".

Prop (unique continuation) $u_0, u_1 : (\Sigma, j) \to (M, J)$ J-hol curve that agree to infinite order at some pt $z_0 \in \Sigma$, then $u_0 \equiv u_1$.

connected

Another way to express this, either u has isolated zero or vanishes identically.

Pf $S = \{ z \in \Sigma \mid u_0 \equiv u_1 \text{ to infinite order} \}$ is closed obviously and non-empty. ($z_0 \in S$)

Near z_0, in local chart D we have for $i = 0, 1$,

$$\frac{\partial u_i}{\partial s} + J(u_i(z)) \frac{\partial u_i}{\partial t} + \underline{\underline{0}} = 0$$

Then consider $w = u_1 - u_0$, then

$$\frac{\partial w}{\partial s} + J(u_1(z)) \frac{\partial u_1}{\partial t} - J(u_0(z)) \frac{\partial u_0}{\partial t} = 0$$

$$\Leftrightarrow \left(\frac{\partial w}{\partial s} + J(u_1(z)) \frac{\partial w}{\partial t} \right) + \underbrace{\left(J(u_1(z)) - J(u_0(z)) \right) \frac{\partial u_0}{\partial t}}_{(*)} = 0$$

For term (Δ), rewrite it as follows

$$(\Delta) = \int_0^1 \frac{d}{d\tau} J(u_0(z) + \tau \cdot W(z))\, d\tau \cdot \frac{\partial u_0}{\partial t}$$

$$= \int_0^1 \frac{d}{d\tau} J(u_0(z) + \tau \cdot W(z)) \cdot \frac{\partial u_0}{\partial t}\, d\tau \overset{\text{chain}}{\underset{\text{rule}}{=}} B(z) \cdot W(z) \text{ for some } B(z)$$

rescaling of matrices $\frac{d}{d\tau} J(u_0(z) + \tau \cdot W(z))$ by each component in $\frac{\partial u_0}{\partial t}$.

depending on J' and $\frac{\partial u}{\partial t}$.

$$\Rightarrow \frac{\partial W}{\partial s} + J(u_1(z)) \frac{\partial W}{\partial t} + B(z) \cdot W(z) = 0$$

By Carleman Similarity Principle, $\exists\ \Phi : D' (\subset D) \longrightarrow GL(2n, \mathbb{R})$ s.t. (disk near z_0)

$\Phi(z) \cdot W(z)$ is holomorphic on D' and $\Phi \in W^{1,p}$ for every $p < \infty$.

$\overset{\text{Sobolev}}{\underset{\text{emb } p>2}{\Longrightarrow}}$ $\exists\ C$ s.t. $|\Phi(z)| < C$ for all $z \in D'$.

$$\Rightarrow \frac{|\Phi(z) W(z)|}{|z - z_0|^k} \leq C \cdot \frac{|u_1(z) - u_0(t)|}{|z - z_0|^k} \longrightarrow 0 \quad \text{as } z \to z_0$$

by our hypothesis.

$\Rightarrow \Phi(z) W(z) \equiv 0$ on D' and then $W(z) \equiv 0$ (so $u_0 = u_1$ on D').

$\Rightarrow S$ is also open $\Rightarrow S = \Sigma$. $\square$

Prop $u : (\Sigma, j) \longrightarrow (M, J)$ J-hol. (Σ connected) If u is not constant, then crit pts of u is discrete. In particular, if Σ is cpt, then # crit pts is finite.

Pf Take $\frac{\partial}{\partial s}$ to $\frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} = 0 \Rightarrow \frac{\partial^2 u}{\partial s^2} + J(u) \frac{\partial^2 u}{\partial s \partial t} + \frac{\partial J(u)}{\partial s} \frac{\partial u}{\partial t} = 0$

Set $v = \frac{\partial u}{\partial s}$ and we get

$\frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} = 0 \Rightarrow \frac{\partial u}{\partial t} = J(u) v$

$$\frac{\partial v}{\partial s} + J(z) \frac{\partial v}{\partial t} + \frac{\partial J(z)}{\partial s} J(z) \cdot v = 0$$

$\overset{\text{Carleman}}{\underset{\text{Similarity principle}}{\Longrightarrow}}$ $\exists$ invertible Φ s.t. $\Phi \cdot v$ is hol $\overset{\cdot \Phi^{-1}}{\Longrightarrow}$ $v \left(= \frac{\partial u}{\partial s}\right)$ has either isolated zero or vanishes identically

$v = 0 \Leftrightarrow du = 0$

$\Rightarrow$ crit pts are isolated (so discrete) (b/c u is not constant). $\square$

2. local Intersection between J-hol curves.

$u, v : \mathbb{D} (\subset \mathbb{C}) \longrightarrow (M, J)$ J-hol

<u>Question</u>: How do $\mathrm{im}(u)$ and $\mathrm{im}(v)$ intersect each other?

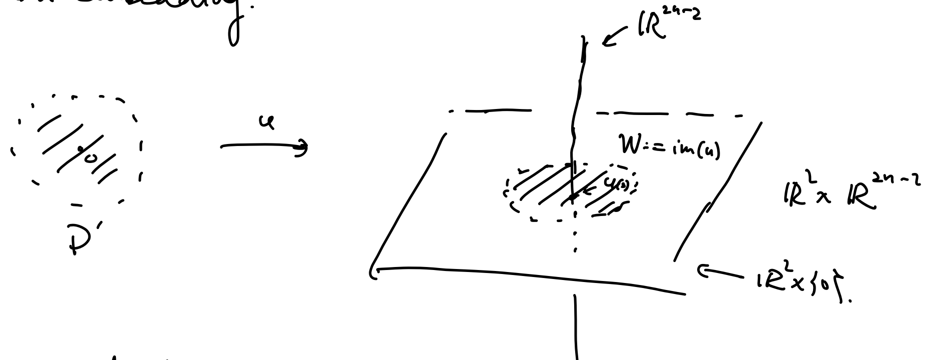
Let's try to make some analysis by ourselves first.

- <u>Assume</u> $du(0) \neq 0$.

Then $du(0) : T_0 D \longrightarrow T_{u(0)} M$ is injective.

Similarly, for z sufficiently close to 0, $du(z) \neq 0 \Rightarrow du(z)$ is injective.

Therefore, by shrinking the NBH of 0 more if necessary, $u : D' (\ni 0) \subset D \to (M, J)$ is an embedding.



Choose a coordinate near $u(0)$ s.t. $\mathrm{im}(u) \subset \mathbb{R}^2 \times \{0\}$, and one can even make $\mathbb{R}^2$ in $\mathbb{R}^2 \times \{0\}$ in a complex nature: $J|_{\mathbb{R}^2 \times \{0\}} : \mathbb{R}^2 \times \{0\} \circlearrowleft$.
(b/c u is J-hol)

2×2 on $\mathbb{R}^2 \times \{0\}$

$$\Longrightarrow \quad J = \begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix} \quad \underset{\text{basis}}{\overset{\text{change}}{\rightsquigarrow}} \quad J = \begin{pmatrix} \ast & 0 \\ 0 & \ast \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

$(2n-2) \times (2n-2)$ on $\{0\} \times \mathbb{R}^{2n-2}$.

Under this coordinate, $u: D \longrightarrow (M, J)$ can be locally written as
depending on u

$$u = (u_1, 0) \quad \text{where } u_1 : D \to \mathbb{C}$$

- Consider $\pi : \mathbb{C} \times \mathbb{C}^{n-1} \longrightarrow \mathbb{C}^{n-1}$ projection onto the second factor.

Note that $\forall\ (x, \tilde{x}) \in \mathbb{C} \times \mathbb{C}^{n-1},\ w \in W$

$$\pi(J(w,0)(x,\tilde{x})) = \pi\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\underset{\substack{\uparrow \\ \text{at this pt}}}{(w,0)}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix}\right) = \pi\left(A(w,0)x,\ D(w,0)\tilde{x}\right)$$
$$= D(w,0)\tilde{x}$$

Consider another J-hol curve $v: D \longrightarrow (M, J)$, under the coordinate above,

$$v \underset{\substack{\uparrow \\ \text{locally}}}{=} (v_1, \tilde{v}) \quad \text{where } v_1 : D \to \mathbb{C},\ \tilde{v} : D \to \mathbb{C}^{n-1}.$$

Then from $0 = \frac{\partial v}{\partial s} + J(v(z))\frac{\partial v}{\partial t}$, we get

$$0 = \pi_*\left(\underset{dv(\partial/\partial s)}{\left(\frac{\partial v_1}{\partial s} + \frac{\partial \tilde{v}}{\partial s}\right)} + J(v_1(z), \tilde{v}(z))\frac{\partial v}{\partial t}\right)$$

$$= \frac{\partial \tilde{v}}{\partial s} + \pi_*\left(J(v_1(z), 0)\frac{\partial v}{\partial t} + \left(J(v_1(z), \tilde{v}(z)) - J(v_1(z), 0)\right)\frac{\partial v}{\partial t}\right)$$

$$= \frac{\partial \tilde{v}}{\partial s} + \pi_*\left(J(v_1(z), 0)\left(\frac{\partial v_1}{\partial t} + \frac{\partial \tilde{v}}{\partial t}\right) + \int_0^1 \frac{d}{d\tau} J(v_1(z), \tau\tilde{v}(z))\frac{\partial v}{\partial t}\, d\tau\right)$$

$$= \frac{\partial \tilde{v}}{\partial s} + D(v_1(z), 0)\frac{\partial \tilde{v}}{\partial t} + B(z)\cdot\tilde{v}(z)$$

This is an almost complex str on $\mathbb{R}^{2n-2}$

$B : D \to \mathrm{End}(\mathbb{R}^{2n-2})$.

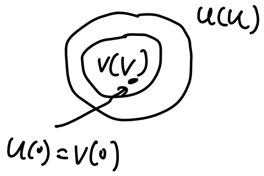
- Apply the argument of Prop (unique continuation), then we get $\exists D' \subset D$ s.t. $\tilde{v}$ is either vanishing at single pt $0 \in D'$ or $\equiv 0$ over D'.

<u>Prop A</u> $u, v : D (\subset \mathbb{C}) \longrightarrow (M, J)$ J-hol and $u(0) = v(0)$, $\underline{du(0) \neq 0}$.
crucial assumption.

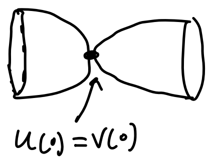
Then one of the following holds:

(1) $\exists$ NBH $\overset{(0\in)}{\mathcal{U}}, V \subset D$ s.t. $v(V) \subset u(\mathcal{U})$. In particular, $\exists$ a holomorphic map $\varphi : V \to \mathcal{U}$ s.t. $v|_V = u \circ \varphi$

(2) $\exists$ NBH $\overset{(0\in)}{\mathcal{U}, V} \subset D$ s.t. $u(\mathcal{U}) \cap v(V) = \{u(0)\}$



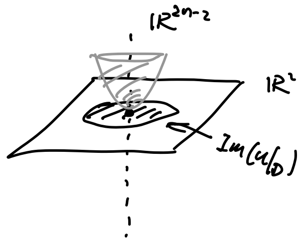
case (1)



case (2)

<u>Pf</u>. Under the coordinate discussed above,

$$u = (u_1, 0) \quad \text{and} \quad v = (v_1, \tilde{v})$$



If $\tilde{v}|_{D'}$ vanishes only at 0, then

$$u(D) \cap v(D') = \{0\} = \{u(0)\}.$$

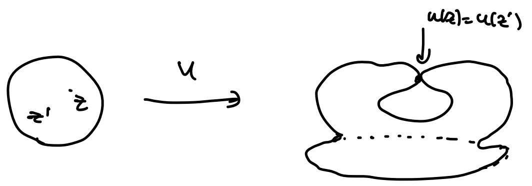
If $\tilde{v}|_{D'}$ vanishes identically on D', then over D', $v = (v_1, 0)$.

Now, things are reduced to 2-dim'l situation $u_1, v_1 : D' \longrightarrow \mathbb{C}$.

Since $u_1(0) = v_1(0)$, $\exists D'' \subset D'$ s.t. $v_1(D'') \subset u_1(D')$ $(\Rightarrow v(D'') \subset u(D'))$.

Then the desired holomorphic map $\varphi := u_1^{-1} \circ v_1$ (since u_1 is emb). $\square$

The intersection could be more complicated!



a self-intersection pt

For J-hol curve $u : (\Sigma, j) \to (M, J)$, denote ← Here, not nec cpt

$\Delta(u) = \{ z \in \Sigma \mid \exists z' \in \Sigma$ ($\neq z$), D, D' NBH of z, z' respectively, s.t. $u(z) = u(z')$ and $u(D) \cap u(D') = u(z) \}$

<u>Prop B'</u> When Σ is cpt, $\Delta(u)$ is finite

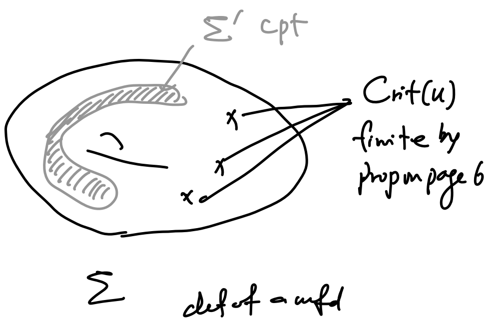
(Hard version)

<u>Prop B</u> when Σ is cpt, $\Delta(u)$ is at most countable and if any sequence of pts in $\Delta(u)$ accumulates, the limit must be in $\mathrm{Crit}(u)$.

<u>Rmk</u> Prop B $\longrightarrow$ Prop B', one needs to show that pts in $\Delta(u)$ will NOT accumulate at $\mathrm{Crit}(u)$. This is from a result by Micallef-White.

<u>Pf of Prop B</u>:

- By def, $\Delta(u)$ is a discrete subset of Σ. ↙ Note that a discrete subset in a cpt space may <u>not</u> be finite! (A discrete & cpt subset is finite).



For $\forall \Sigma' \subset \Sigma \setminus \mathrm{Crit}(u)$ cpt we aim to show $\Delta(u) \cap \Sigma'$ is closed (*) ($\Rightarrow \Delta(u) \cap \Sigma'$ is cpt $\Rightarrow \Delta(u) \cap \Sigma'$ is finite)

$$\Sigma \setminus \mathrm{Crit}(u) = \underbrace{\bigcup}_{\text{countably many}} \Sigma' \quad (\text{def of a mfd}; \ \Sigma' \text{ cpt}) \Rightarrow \Delta(u) = \underbrace{\left(\Delta(u) \cap (\Sigma \setminus \mathrm{Crit}(u))\right)}_{\substack{\text{countable} \\ = \text{countable} \times \text{finite.}}} \cup \underbrace{\left(\Delta(u) \cap \mathrm{Crit}(u)\right)}_{\text{finite.}}$$

$\Rightarrow$ desired conclusion

note that $z \in \Sigma'$ automatically since Σ' is cpt (so closed)

- Verify ($\clubsuit$):

Suppose $\{z_m (\in \Delta^{(u)} \cap \Sigma)\}_m \longrightarrow z$, then we need to show $z \in \Delta^{(u)} \cap \Sigma'$ as well.

By def, $\exists \{w_m\} \subset \Sigma$ (corresponding to $\{z_m\}$) s.t. $u(z_m) = u(w_m)$

Suppose $w_m \longrightarrow w \in \Sigma$, so by continuity $u(z) = u(w)$.

<u>KEY</u>: Since $z \notin \mathrm{Crit}(u)$, u is an embedding near z.
(in particular homeomorphism)

$\Longrightarrow$ $z \neq w$ (otherwise z_m, w_m lie in the same NBH of z
and then $u(z_m) = u(w_m) \Rightarrow z_m = w_m$, contradicting
to the defining property of $\Delta^{(u)}$).

Then Prop A applies for $\underset{\text{NBH of } z}{\underset{\cup}{u: D}} \longrightarrow \mathbb{C}^n$ and $\underset{\text{NBH of } w}{\underset{\cup}{v: D}} \longrightarrow \mathbb{C}^n$ s.t.

either $u(D) \cap v(D) = \{u(z) = u(w)\}$ $\Rightarrow$ $z \in \Delta(u)$

or $v(D') \subset u(D)$ (impossible by defining property of Δ).
$\uparrow$ for smaller D' if nec.

- The argument above also shows that if $\{z_m\}$ in $\Delta^{(u)}$ converges, then the limit has to be in $\mathrm{Crit}(u)$ (cf. KEY above). $\square$

Prop A + Prop B' (+ another top lemma) imply the following result.
This result reduces the topology of a J-hol curve, but traded with "singular points"