5. Revisit Morse theory

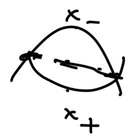
$(M, g, F) \longrightarrow \mathrm{Crit}(F) =$ generators of Morse chain cpx + grading (via Hessian of F)

F: Morse fcn

$\longrightarrow \partial_{\mathrm{Morse}}$ is defined by studying or counting moduli space $\mathcal{M}(x_-, x_+)$

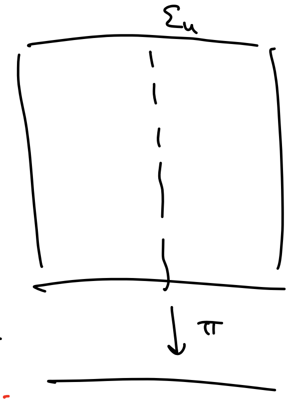
$$\mathcal{M}(x_-, x_+) = \{ u : \mathbb{R}_s \to M \mid \dot{u} = -\nabla_g F(u), \ \lim_{s \to \pm\infty} u(s) = x_\pm \}$$

<u>Classical approach</u>: $\mathcal{M}(x_-, x_+) = W^u(x_-) \cap W^s(x_+)$ under Morse-Smale condition



$\Rightarrow \dim \mathcal{M}(x_-, x_+) = \mathrm{ind}_{\mathrm{Morse}}(x_-) - \mathrm{ind}_{\mathrm{Morse}}(x_+)$.

<u>New approach</u> (Schwarz 93 - Morse homology book) ← inspired by Floer theory.



$\sigma = \frac{\partial}{\partial t} + \nabla_g F$

(one can also write this in terms of 1-form-valued operator)
(then $\mathcal{E}_u = W^{k-1,p}(\Omega^1(\mathbb{R}, u^*TM))$)

$\mathcal{E} = \bigsqcup_u \{u\} \times W^{k-1,p}(u^*TM)$ ← upgrade of $\Gamma(u^*TM)$

$\downarrow$

$\mathcal{B} = W^{k,p}(\mathbb{R}, M)$ ← upgrade of $C^\infty(\mathbb{R}, M)$

$\Rightarrow \mathcal{M} = \{ u \in \mathcal{B} \mid u \in \sigma^{-1}(0) \}$ + asymptotic ends

$\Rightarrow D_u =$ linearization of σ at u: $W^{k,p}(u^*TM) \longrightarrow W^{k-1,p}(u^*TM)$

By computation, $\forall \xi \in W^{k,p}(u^*TM)$,

$$D_u(\xi) = \nabla \xi + \nabla_\xi \nabla_g F(u)$$

$$\overset{\text{locally}}{=} \left(\frac{\partial}{\partial s} + A(s)\right)(\xi) \quad \text{for an } \mathbb{R}\text{-family of operator } A(s)$$

(as symmetric matrices)

Then one aims to show that D_u is a Fredholm operator (and conclude $\mathcal{M}(x_-, x_+)$ is a mfd with correct dim).

Rmk Compared with discussion above, here are two new points:

(b/c asymptotic ends)

1. the domain is $\mathbb{R}$, non-cpt, so we need more analysis to take care of the asymptotic ends.

$$A(s) \longrightarrow \underbrace{A_\pm \in \overset{n\times n}{\text{matrix}} \; (= \text{Hess}(f) \text{ at } x_\pm}_{\text{geometry}})$$

In most discussion above, Σ is almost assumed to be closed.

2. $$\dim \mathcal{M}(x_-, x_+) = \text{ind}(D_u)$$

$$= \text{expression in terms of } A_\pm.$$

one should expect this (see SFT 5 for more precise calculation of $\text{ind}(D_u)$)

Thm (prop 3.1 in [Wen]) $\text{ind}(D_u) \overset{(*)}{=}$ # negative eigenvalues of A_-

$-$ # negative eigenvalues of A_+

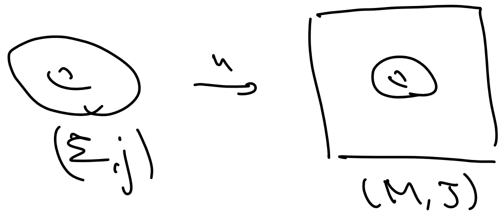
$$\overset{\text{by geometry}}{=} \text{ind}_{\text{Morse}}(x_-) - \text{ind}_{\text{Morse}}(x_+)$$

In (*), we can express it in a more unified way that applies to even ∞-dim'l setting:

via spectral flow next section

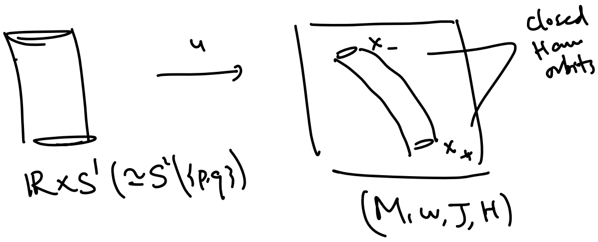
$$\# \text{ neg eigenvalue of } A^- - \# \text{neg eigenvalue of } A^+ = \mu^{\text{spec}}(A_-, A_+)$$

Here is a short summary:

① 

For generic J, $\mathcal{M}_J = \mathcal{M}_{J,A} \cup \mathcal{M}_{J,B} \cdots$ is a mfd

$\dim \mathcal{M}_{J,A} = \operatorname{ind}(D_u) +$ some ind from A.

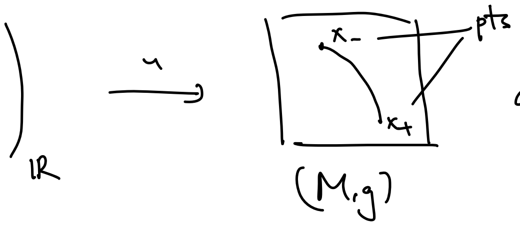
② 

For generic J, --

$\dim \mathcal{M}_{J,A,x_-,x_+} = \operatorname{ind}(D_u) +$ some ind from A

$=$ some ind from A + ind difference between asymptotic operators at $x_\pm$

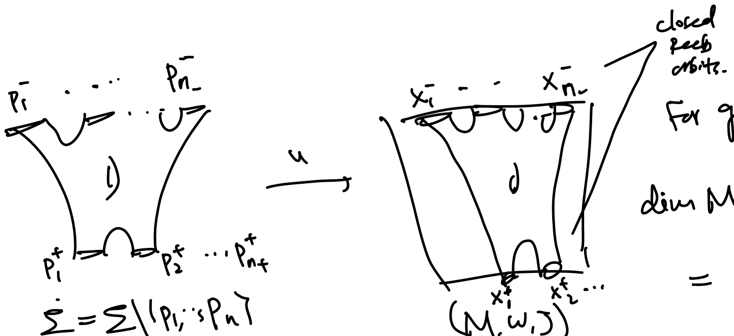
1-dim'l lower analogue (Morse)



For generic g. . . .

$\dim \mathcal{M}_{g,x_-,x_+} = \operatorname{ind}(D_u) +$ some ind from x_-, x_+

$= \operatorname{ind}_{Morse}(x_-) - \operatorname{ind}_{Morse}(x_+)$

③ 

$\dot{\Sigma} = \Sigma\setminus\{p_1, \cdots, p_n\}$

$= \Sigma\setminus\{p_1^-, \cdots, p_{n_-}^-, p_1^+, \cdots, p_{n_+}^+\}$

where $n_- + n_+ = n$

symp cobordism.

For generic J ...

$\dim \mathcal{M}_{J,A,(x_1^-, \cdots, x_{n_-}^-, x_1^+, \cdots, x_{n_+}^+)}$

$= \operatorname{ind}(D_u) +$ some ind of A

$=$ some ind of A + ind difference between asymptotic operator at $x_i^\pm$.

If in Morse theory, the index of the asymptotic operator $A_\pm$ is the Morse index at $x_\pm$, what will be the analogue of some index associated to asymptotic operator at closed Reeb orbits?

Attempt: define a "Morse" fcn $\mathcal{A}$ on loop space of a contact mfd ΛX (when X can be viewed as a contact boundary of a symp cobordism).
and closed Reeb orbits serve as $\mathrm{Crit}(\mathcal{A})$.

Then similarly to the Morse discussion, for a closed Reeb orbit $x_i^{\pm}$

$$A_i^{\pm} \text{ at } x_i^{\pm} \simeq \text{Hessian of } \mathcal{A} \text{ at } x_i^{\pm}.$$

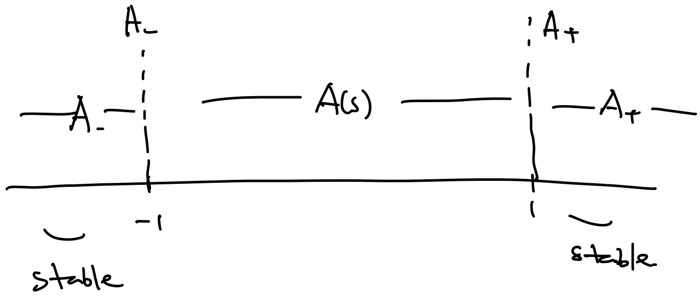
<u>Rmk</u> An essential difference: in Morse case, $A^{\pm}$ (at $x_{\mp}$) is a single matrix.
in Reeb case, $A_i^{\mp}$ (at $x_i^{\mp}$) is a loop of matrices

cf. Ex 3 in Homework One, one gets (under a trivialization along $x_i^{\pm}$)

$$A_i^{\pm} = -J\frac{d}{dt} - S_i^{\pm}(t). \quad \text{for a loop of symmetric matrices } S(t).$$

Though they are in different formulation, $\exists$ a uniform perspective.

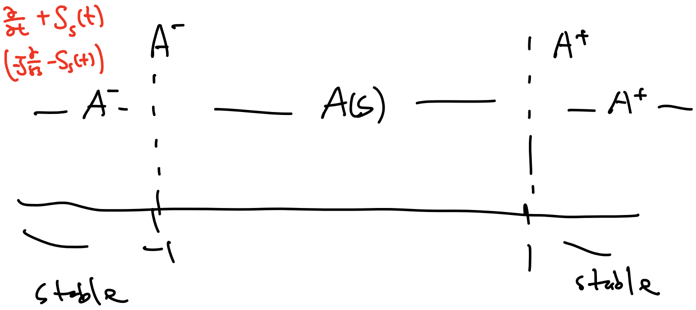
Morse case:



each $A(s)$ is a sym. matrix
$A(s): \mathbb{R}^n \circlearrowleft$

Reeb case: $Du = \frac{\partial}{\partial s} + J\frac{\partial}{\partial t} + S_s(t)$
$= \frac{\partial}{\partial s} - (J\frac{\partial}{\partial t} - S_s(t))$

(for brevity, only one positive end and one neg. end)



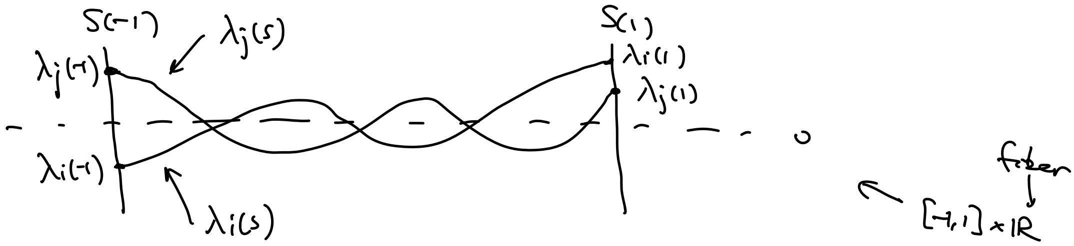
each $A(s)$ is a sym. operator
$A(s): W^{1,2}(S^1; \mathbb{R}^{2n}) \to L^2(S^1; \mathbb{R}^{2n})$

6. Spectral flow

In finite dim'l setting, given a continuous family of matrices:

$$S = S(s) : [-1,1] \longrightarrow \mathrm{Sym}(n)$$

one can keep recording eigenvalues of $S(s)$ along $[-1,1]$ with multiplicity, and $\exists$ n-many continuous path $\lambda_j : [-1,1] \to \mathbb{R}$ tracing eigenvalues of $S(s)$.

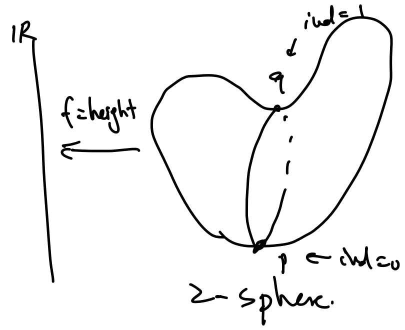


Then

$$\mu^{spec}(S) := \#\{ j \in \{1,\cdots,n\} \mid \lambda_j(-1) < 0 < \lambda_j(1) \} - \#\{ j \in \{1,\cdots,n\} \mid \lambda_j(-1) > 0 > \lambda_j(1) \}$$

eigenvalues crossing the value 0.

Ex

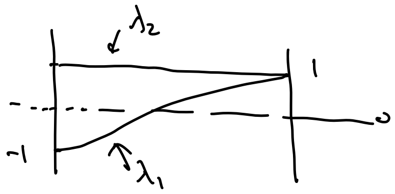


geometric model

$$S(s) = \begin{pmatrix} s & \\ & 1 \end{pmatrix}$$ constant

$s = -1 \Rightarrow S(-1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{Hess}(f)(q)$

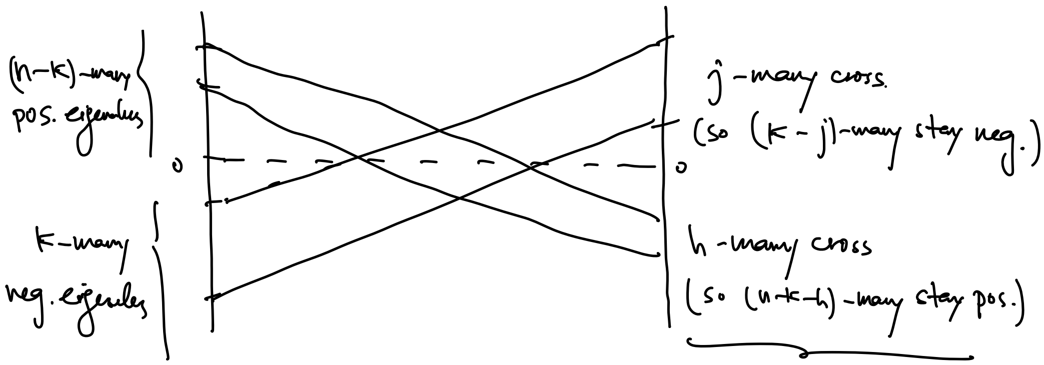
$s = 1 \Rightarrow S(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{Hess}(f)(p)$



$\Rightarrow \mu^{spec}(S) = 1 - 0 = 1$

Ex Suppose $S(-1)$ and $S(1)$ are both invertible, then ← geometrically $x^{\pm}$ are crit pts for Morse fcn.

$$\mu^{spec}(S) = \#\text{ neg. eigenvalues of } S(-1) - \#\text{ neg. eigenvalues of } S(1).$$



$\#$ neg eigenvalue $= k-j+h$.
$\#$ pos. eigenvalue $= j+n-k-h$

$$\Rightarrow \quad \mu^{spec}(S) = j - h$$
$$= k - (k-j+h)$$
$$= \#\text{ neg eigenvalue of } S(-1) - \#\text{ neg eigenvalue of } S(1) .$$

Rmk.

- Choice of $\{\lambda_j\}_{j\in\{1,\dots,n\}}$ is not unique, but $\mu^{spec}(S)$ is independent of such choice (b/c different choices are related by a homotopy).
- The crucial properties of S (that ensures the $\exists$ of $\{\lambda_j\}$) are spectrum of $S(s)$ are real and discrete ← obvious in finite-dim'l setting.

For ∞-ly dim'l setting: $A(s)$ a L^2-symmetric Fredholm operator of ind $=0$ $W^{1,2}(S^1, \mathbb{R}^{2n}$ $\mathbb{C}^n) \to L^2(S^1, \mathbb{R}^{2n}$ $\mathbb{C}^n)$ (sometimes, $\mathbb{R}^{2n}$ is replaced by $\mathbb{C}^n$), by deep spectral theory of (unbounded) self-adjoint Fredholm operator (of ind 0):

<u>Thm</u> $\mathrm{Spec}(A(s)) := \{ \eta \in \mathbb{C} \mid A - \eta \cdot I \text{ is NOT invertible} \}$

is a discrete subset in $\mathbb{R}$. Here $I: W^{1,2}(S^1, \mathbb{C}^n) \hookrightarrow L^2(S^1, \mathbb{C}^n)$ inclusion.

Then, for a family of

$$A = \underset{\substack{\uparrow \\ s \in [-1,1]}}{A(s)} := -J \frac{\partial}{\partial t} - \underset{\substack{\in \mathrm{Map}(S^1, \mathrm{Sym}(2n)) \\ \swarrow}}{S(t)} : W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$$

one defines

$$\mu^{spec}(A) := \#\{ j \in \mathbb{Z} \mid \lambda_j(-1) < 0 < \lambda_j(1) \} - \#\{ j \in \mathbb{Z} \mid \lambda_j(-1) > 0 > \lambda_j(1) \}$$

Note that

e.g. $A = -J \frac{\partial}{\partial t}$ (with $S \equiv 0$)

- Here, there could ∞-ly many $\{\lambda_j\}_{j \in \mathbb{Z}}$, pos./neg eigenvalues.

The reality is that there are only finitely many eigenvalues crossing 0.

- One often define $\mu^{spec}(A)$ under the condition that both $\mathrm{Ker}(A(-1))$ and $\mathrm{Ker}(A(1))$ are trivial. (cf. non-deg in Section 4 in this lecture)

Ex/Def. (as an application of μ^{spec} in this setting)

Take $A^+ = -J \frac{\partial}{\partial t} - S$ for some constant loop $S \in \mathrm{Map}(S^1, \mathrm{Sym}(2n))$ that is similar to $\begin{pmatrix} \underbrace{1 \ \ddots \ 1}_{n} & \\ & \underbrace{-1 \ \ddots \ -1}_{n} \end{pmatrix}$, then for any given $A (= A^-)$, define

in fact ANY

$$\mu_{CZ}(A) := \mu^{spec}(A^-, A^+)$$

This is well-defined in the sense that it's ind of S and $A(s)$ connecting A^- and A^+.