Therefore, in order to prove $\mathcal{M}_J = \left\{ \begin{matrix} \text{moduli space} \\ \text{of } J\text{-hol curve} \end{matrix} \right\} = \bar\partial_J^{-1}(0)$ is a mfd, we just need to show $\forall u \in \bar\partial_J^{-1}(0)$, D_u is surjective. ← we actually arrive at this pt by an informal argument near the end of SFT-2.

3. Remarks

Rmk If so, then

$$\dim \mathcal{M}_J = \dim \ker(D_u) \qquad \forall u \in \bar\partial_J^{-1}(0)$$

One ambiguity: $\bar\partial_J^{-1}(0)$ may not be (path) connected!

$$\bar\partial_J^{-1}(0) \underset{\substack{\uparrow \\ \text{one way to decompose}}}{=} \bar\partial_{J,A}^{-1}(0) \sqcup \bar\partial_{J,B}^{-1}(0) \cup \cdots$$

A, B htp class of $\mathrm{im}(u)$ in $H_2(M;\mathbb{R})$ (possibly with asymptotic ends/or boundary conds).

$\Rightarrow$ one consider $\mathcal{M}_{A,J}$ (i.e. fix J but with further top constraints)

and $\dim \mathcal{M}_{A,J}$ should depend on class A (later lectures).

Moreover, in each connected component, say $\bar\partial_{J,A}^{-1}(0)$,

$$\{u_t\}_{t\in[0,1]} \rightsquigarrow D_{u_t} \Rightarrow \text{ind}(D_u) \text{ is ind of the path}$$

$\Rightarrow$ a benefit to calculate $\text{ind}(D_u)$ from some "special" $u \in \bar\partial_{J,A}^{-1}(0)$.

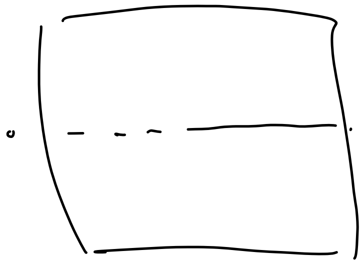
Rmk (a delicate point). Does $\text{ind}(D_u)$ (or more precisely $\ker(D_u)$ or $\text{coker}(D_u)$) depends on the regularity degree k? NO

$\underbrace{J \text{ is smooth}}_{\substack{\text{assumption} \\ \text{and } u \text{ is } J\text{-hol.}}} \xRightarrow[\text{last time}]{\text{thm}} u \in W^{k,p}(\Sigma, M) \cap C^1$ is in fact smooth $\Rightarrow$ thm above applies for any $k \geq 1$.

$\Rightarrow \xi \in \ker(D_u) \subset W^{k,p}$ is in fact smooth.

Rmk The reality is that for J, D_u is not nec surjective :c.

To deal with a general case, consider



$\mathcal{E}$

Some space of J (as a Banach mfd)

$\bar\partial(u,J)$

now J is a new variable

$$\downarrow \pi$$

$$\mathcal{B} = W^{k,p}(\Sigma, M) \times \mathcal{J}$$

$$\mathcal{M}^{univ} = \{ (u, J) \mid \bar\partial_J u = 0 \} \xrightarrow[p]{\text{projection}} \mathcal{J}$$

Then $D_{u,J}$ = linearization of $\bar\partial$ at (u,J) : $W^{k,p}(u^*TM) \times T_J\mathcal{J} \to T^{vert}_{(u,J)}\mathcal{E}$.

and can be expressed as $D_{u,J} = D_u$ + extra term from $T_J\mathcal{J}$.

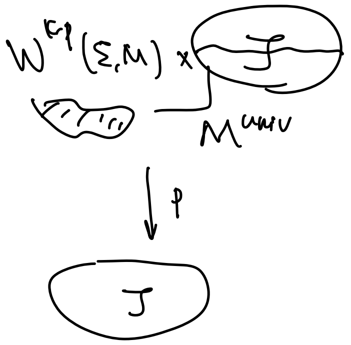
D_u ↑ the one discussed in SFT2

(more precisely, $(D_u, 0) + (0, \text{extra term})$)

For each $J \in \mathcal{J}$, since D_u is Fredholm (in particular, $\operatorname{coker}(D_u)$ is finite-dimensional), when $T_J\mathcal{J}$ is sufficiently large, $D_{u,J}$ will be surjective.

In general, this needs extra care due to "multiple covers."

$\Rightarrow$ $D_{u,J}$ is also Fredholm and we get $\mathcal{M}^{univ}$ as a smooth mfd.



$\Rightarrow$ $W^{k,p}(\Sigma, M) \times \mathcal{J}$

$\mathcal{M}^{univ}$

$\downarrow p$

$\mathcal{J}$

$\exists$ a lot by Smale-Sard's thm.

check if $J \in \mathcal{J}$ is a regular value of projective π_J, then

$$p^{-1}(J) = \mathcal{M}_J \subset \text{▨} \times J$$

is a smooth mfd.

4. Fredholm from matrices

Recall $Sp(2n) = \{ X \in M_{2n\times 2n}(\mathbb{R}) \mid X^T J X = J \}$ where

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \quad \text{(in coordinate } (x_1, \cdots, x_n, y_1, \cdots, y_n))$$

Its Lie algebra $sp(2n)$ computes as

$$sp(2n) = \{ B \in M_{2n\times 2n}(\mathbb{R}) \mid B^T J + JB = 0 \}$$

Observe that $\underbrace{-JB}_{\text{for } B \in sp(2n)}$ is symmetric b/c $(-JB)^T = -B^T J^T = B^T J = -JB$ and vice versa.

$\Longrightarrow$ $\Psi = \{\Psi(t)\}_{t\in[0,1]}$ a path in $Sp(2n)$ s.t. $\Psi(0) = \mathbb{1}$.

then

$$S(t) := -J \cdot \Psi'(t) \cdot \Psi(t)^{-1} \quad \swarrow \text{ this is an ODE.} \quad (*)$$

gives a path $S = \{S(t)\}_{t\in[0,1]}$ a path in $Sym(2n)$

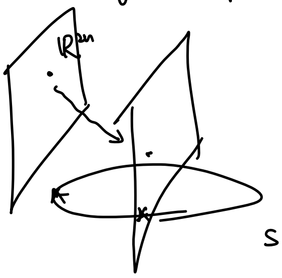
(Conversely, given $S(t)$, we can recover $\Psi(t)$ by solving the ODE above).

$\Longrightarrow$ $\left\{ \begin{array}{c} \{\Psi(t)\}_{t\in[0,1]} \text{ in } Sp(2n) \\ \text{starting from } \mathbb{1} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \{S(t)\}_{t\in[0,1]} \text{ in } Sym(2n) \right\}$

$(*) \Longleftrightarrow -J\Psi'(t) - S(t)\Psi(t) = 0 \Longleftrightarrow \left(-J\frac{d}{dt} - S(t)\right) \cdot \Psi(t) = 0$

view multiplication by a matrix as an action

Rmk How to get a path of matrices?



$S^1\times\mathbb{R}^{2n}$ ⟲ $s\in\Gamma(S^1, S^1\times\mathbb{R}^{2n})$ $\Longrightarrow$ $s(p), s(q)\in\mathbb{R}^{2n}$ and $\exists$ matrix $A_{p,q}$ s.t. $A_{p,q}\cdot s(p)=s(q)$.
(then fix $p=0$ and move $q\in[0,1]$)

Therefore, we can generalize from paths of matrices to fcn space $S^1\to\mathbb{R}^{2n}$ (or $[0,1]\to\mathbb{R}^{2n}$).

both are Hilbert space

Introduce the notation: $A: W^{1,2}(S^1,\mathbb{R}^{2n}) \longrightarrow L^2(S^1,\mathbb{R}^{2n})$

$$f=f(t)\longmapsto -J\frac{d}{dt}\cdot f(t) - S(t)\cdot f(t) =: A(f)$$

(i.e. $A := -J\frac{d}{dt} - S$, where $S: S^1\longrightarrow \mathrm{Sym}(2n)$)

Ex Let's play with operator A, when S in A is induced by a path $\Psi=\{\Psi_t\}_{t\in[0,1]}$ in $Sp(2n)$.

in the sense that via the ODE above, $S(t)$ satisfies $S(0)=S(1)$ (so defined over S^1).

* To get such Ψ, one can consider $\Psi=\{\Psi(t)\}_{t\in\mathbb{R}}$ satisfying $\Psi(t+1)=\Psi(t)\cdot\Psi(1)$

- $f\in\ker(A)$, by def, $J\frac{df}{dt} = -S(f)$, which is equ to

$$\frac{df}{dt} = \Psi'(t)\Psi^{-1}(t)\, f$$

$$= -\Psi(t)\left(\Psi^{-1}(t)\right)'(f)$$

$\Psi(t)\Psi^{-1}(t)=\mathbb{1}$
$\Rightarrow \Psi'(t)\Psi^{-1}(t) = -\Psi(t)\left(\Psi^{-1}(t)\right)'$

$$\Longrightarrow \Psi(t)^{-1}\frac{df}{dt} = -\left(\Psi^{-1}(t)\right)'(f) \iff \frac{d}{dt}\left(\Psi(t)^{-1}\cdot f\right)=0$$

$$\Longrightarrow \Psi(t)^{-1}\cdot f(t) = f(0)\cdot\Psi^{-1}(0) = f(0).$$

$$\Longrightarrow f(t) = \Psi(t)\cdot f(0)$$

Since f is defined over S^1, we have.

$$f(0) = f(1) = \Psi(1)\cdot f(0) \iff \underset{\mathbb{R}^{2n}}{f(0)} \in \ker(\Psi(1) - \mathbb{1}).$$

- Moreover, if $f(0) = 0$, then

$$f(t) = \Psi(t)\cdot 0 = 0 \implies f \equiv 0 \text{ in } W^{1,2}.$$

- Suppose $\exists\ v \in \ker(\Psi(1) - \mathbb{1})$, then define

$$f(t) := \Psi(t)\cdot v \qquad (\text{so } f(0) = v)$$

Then $f(1) = \Psi(1)\cdot v = v = f(0) \implies f \in W^{1,2}(S^1, \mathbb{R}^{2n})$.

Moreover, one can check as in the first bullet above, $A(f) = 0$.

All together one gets

$$\underset{\text{more analysis}}{\ker(A)}\left(\subset W^{1,2}(S^1,\mathbb{R}^{2n})\right) \underset{\text{as a vector space}}{\simeq} \underset{\text{more geometry}}{\ker(\Psi(1) - \mathbb{1})}$$

<u>Rmk</u>. A path $\Psi = \{\Psi(t)\}_{t\in[0,1]}$ in $Sp(2n)$ is called <u>non-degenerate</u> if

$\ker(\Psi(1) - \mathbb{1}) = 0 \quad \left(\underset{\text{Ex above}}{\iff} \ker(A) = 0\right).$

This can be used to define a closed Ham orbit is non-deg.
(cf. Arnold conjecture in SFT-1)

<u>Ex</u> Here is another basic observation of A. For $f, g \in W^{1,2}(S^1, \mathbb{R}^{2n})$,

$$\langle Af, g\rangle = \int_0^1 \langle -J\tfrac{df}{dt} - S(f), g\rangle\, dt$$

inner product in $\underline{\underline{L^2(S^1,\mathbb{R}^{2n})}}$ not $W^{1,2}$!

$\langle , \rangle = \omega(-, J-)$

$$= -\int_0^1 \omega_{std}\left(J\tfrac{df}{dt}, Jg\right) dt - \int_0^1 \langle S(f), g\rangle dt$$

$$= -\int_0^1 \omega_{std}\left(\frac{df}{dt}, g\right) dt - \int_0^1 \langle S(f), g\rangle dt$$

$\int_0^1 \frac{d}{dt}\omega_{std}(f,g)dt = 0$ b/c f, g defined on S^1.

$$= \int_0^1 \omega_{std}\left(f, \frac{dg}{dt}\right) dt - \int_0^1 \langle f, Sg\rangle dt.$$

b/c S is symmetric.

$$= \cdots = \langle f, Ag\rangle$$

$\Rightarrow$ A is symmetric w.r.t $\langle\,,\,\rangle_{L^2}$ on $W^{1,2}(S^1, \mathbb{R}^{2n})$.

<u>Thm</u> Operator $A: W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$ is a Fredholm operator.

<u>Pf</u> We aim to use the useful lemma above to show this: $\exists C>0$ s.t. $\forall f \in W^{1,2}(S^1, \mathbb{R}^{2n})$, we have

$$\|f\|_{W^{1,2}} \leq C\left(\|Af\|_{L^2} + \|f\|_{L^2}\right)$$

The expected cpt operator is $W^{1,2} \hookrightarrow L^2$ (which is cpt, see SFT-3)

Let's do it:

$$\|f\|^2_{W^{1,2}} = \|f\|^2_{L^2} + \|\tfrac{d}{dt} f\|_{L^2}$$

$$= \|f\|^2_{L^2} + \|J(A+S)(f)\|_{L^2}$$

$$= \|f\|^2_{L^2} + \|(A+S)(f)\|_{L^2}$$

$\int_0^1 \langle Jf, Jg\rangle dt$
$= \int_0^1 \omega_{std}(Jf, J(Jg))dt$
$= \int_0^1 \omega_{std}(g, Jf)dt = \int_0^1 \langle g, f\rangle dt.$

$$= \|f\|^2_{L^2} + \|Af\|^2_{L^2} + 2\langle A(f), S(f)\rangle + \|Sf\|^2_{L^2}$$

Cauchy-Schwartz inequality $\langle Af, Sf\rangle \leq \|Af\|^{1/2}\|Sf\|^{1/2}$ and $2ab \leq a^2+b^2$

$$\leq \|f\|^2_{L^2} + 2\|Af\|^2_{L^2} + 2\|Sf\|^2_{L^2}$$

$$\leq C\left(\|Af\|^2_{L^2} + \|f\|^2_{L^2}\right)$$ b/c S is a bounded operator

$$\leq C\left(\|Af\|_{L^2} + \|f\|_{L^2}\right)^2$$

(a continuous family of matrix over a closed interval $[0,1]$).

So A is semi-Fredholm (closed image and $\underline{\ker(A)}$ finite-dim'l).

already known if $S = S_{\Xi}$ by the first Ex above in this section.

Now, Let's prove dim(coker A) is also finite-dim'l. In fact, we will show that

$$\dim(\operatorname{coker} A) = \dim(\ker A) \qquad (**)$$

For $g \in \operatorname{coker} A$ = orthogonal complement of A in $L^2(S^1, \mathbb{R}^{2n})$, so $\langle w, Af \rangle = 0$

$\forall f \in W^{1,2}(S^1, \mathbb{R}^{2n})$ and then

$$0 = \langle g, -Af \rangle = \langle g, J\tfrac{df}{dt} + Sf \rangle = \langle g, J\tfrac{df}{dt} \rangle + \langle g, Sf \rangle$$

$$= -\langle Jg, \tfrac{df}{dt} \rangle + \langle Sg, f \rangle$$

So $\quad \langle Jg, \tfrac{df}{dt} \rangle = \langle Sg, f \rangle \iff -\langle J\underline{\tfrac{d}{dt}}g, f \rangle = \langle Sg, f \rangle$

$\Leftarrow$ in a weak sense

$$\Rightarrow \langle \left(-J\tfrac{d}{dt} - S\right)g, f \rangle = 0$$

$$\Rightarrow \left(-J\tfrac{d}{dt} - S\right)g = 0.$$

So $g \in W^{1,2}(S^1, \mathbb{R}^{2n})$ and $g \in \ker A$. Therefore

$$\operatorname{coker} A \subset \ker A \quad (\text{inside } W^{1,2})$$

Since A is symmetric, we also have. $\ker A \subset \operatorname{coker} A \Rightarrow \ker A = \operatorname{coker} A$

$\square$.

recall weak derivative means $\quad \int g f' = -\int \square f$.

<u>Rmk</u> $(**)$ above implies that $\operatorname{ind}(A) = 0$.