Fredholm analysis

0. Rewrite the past

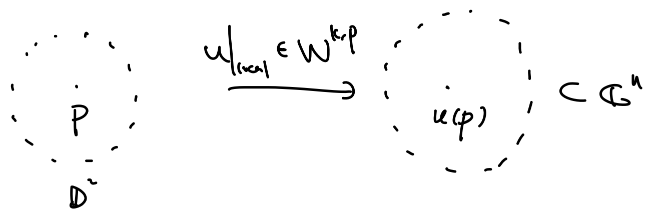
$$\begin{array}{ccc} \mathcal{E} = \bigcup_u \{u\} \times \Omega^{0,1}(\Sigma, u^*TM) & & \mathcal{E} = \bigcup_u \{u\} \times W^{k-1,p}(\Omega^{0,1}(\Sigma, u^*TM)) \\ \bar\partial_J \curvearrowleft \downarrow & \xrightarrow[\text{in terms of Sobolev space}]{\text{upgrade}} & \downarrow \\ \mathcal{B} = C^\infty(\Sigma, M) & & \mathcal{B} = W^{k,p}(\Sigma, M) \end{array}$$

Here, one should view

- $W^{k,p}(\Sigma, M) = \overline{C^\infty(\Sigma, M)}^{\|\cdot\|_{k,p}}$ (similarly to $W^{k-1,p}(\Omega^{0,1}(\Sigma, u^*TM))$)

- $W^{k,p}(\Sigma, M)$ is defined locally from $W^{k,p}(\mathbb{D}^2, \mathbb{C}^n)$.

$\exists$ local chart s.t.



Therefore, the last theorem in SFT-3 can be globally stated:

<u>Thm</u> $u: (\Sigma, j) \to (M, J)$ J-hol and J is C^k, then if $u \in W^{1,p}(\Sigma, M)$ for $p > 2$, then $u \in W^{k,p}(\Sigma, M)$. In particular, if J is smooth, then $u \in C^\infty(\Sigma, M)$.

<u>Rmk</u>: $\bar\partial_J : W^{k,p}(\Sigma, M) \longrightarrow W^{k-1,p}(\Omega^{0,1}(\Sigma, u^*TM))$

$\Longrightarrow$ its linearization (via parallel transport): $(\tilde\nabla \bar\partial_J)(u) : \underset{\text{section}}{W^{k,p}(u^*TM)} \longrightarrow W^{k-1,p}(\Omega^{0,1}(\Sigma, u^*TM))$.

Often one simply the notation by denoting D_u $\left(=(\tilde{\nabla}\bar{\partial}_J)(u)\right)$.

1. Basic Fredholm

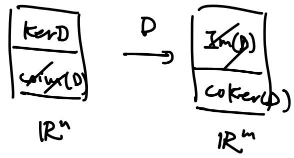
A bounded linear operator $D: \underbrace{X \to Y}_{\text{Banach space (over } \mathbb{R} \text{ or } \mathbb{C})}$ is called a Fredholm operator

if (1) $\mathrm{im}(D)$ is closed in Y; (2) $\dim(\ker D)$ is finite; (3) $\dim\big(\underbrace{Y/\mathrm{im}(D)}_{\text{coker}}\big)$ is also finite.

$$\Longrightarrow \quad \mathrm{ind}(D) = \dim \ker D - \dim \mathrm{coker} D \quad \text{called Fredholm index}$$

<u>Ex</u> $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, then any linear map $D: X \to Y$ is Fredholm.

$$\begin{aligned}
\mathrm{ind}(D) &= \dim \ker D - \dim \mathrm{coker} D \\
&= \dim \ker D - (m - \dim(\mathrm{im} D)) \\
&= n - \dim(\cancel{\text{coim}}(D)) - m + \dim(\cancel{\text{im}}(D)) \\
&= n - m
\end{aligned}$$



Note that this ind(D) is in fact independent of D. In ∞-dim'l this is not true.

<u>Question</u>: How to verify that $D: X \to Y$ is Fredholm?

Hard part: im(D) is closed and dim(kerD) $< \infty$. For coker, one usually take dual.

<u>Lemma</u> (very useful) $D: X \to Y$ b/d linear operator
$K: X \to Z$ cpt operator

If $\exists\, C > 0$ s.t. $\|x\|_X \le C(\|Dx\|_Y + \|Kx\|_Z)\ \forall x \in X$, (could depend on D, K)

then D has a closed image and $\dim(\ker D) < \infty$. ← These two conclusions also supp D to be called semi-Fredholm.

$\Rightarrow$ starting from D in the conclusion, assume $\dim \ker D = n$.

then $\exists\ K_0 : X \to \mathbb{R}^n$ (coming from projection to the complement of $\ker D$) s.t. $K_0 \big|_{\ker D \subset X} : \ker D \simeq \mathbb{R}^n$

$\exists$ of complement comes from Hahn-Banach Thm

$\Rightarrow \quad X \to Y \oplus \mathbb{R}^n \qquad x \mapsto (Dx, K_0 x)$ is injective and has closed image.

(so $\exists$ bijective : $X \to$ image of (D, K_0))

$\xRightarrow[\text{(from open mapping thm)}]{\text{Banach iso thm}}$ the inverse is a bounded operator, i.e. $\exists\ C > 0$

s.t. $\quad \|x\|_X \leq C \left(\|Dx\|_Y + \|K_0 x\|_{\mathbb{R}^n} \right)$

therefore, if $P : X \to Y$ satisfying $\|P\| < \frac{1}{C}$ then

ind of P.

$$\|x\|_X - C\|Px\|_Y \leq C \left(\|(D+P)x\|_Y + \|Px\|_Y + \|K_0 x\|_{\mathbb{R}^n} \right) - C\|Px\|_Y$$

$$\xRightarrow{\|Px\|_Y \leq \|P\|\|x\|_X} \quad \|x\|_X - C\|P\|\,\|x\|_X \leq C \left(\|(D+P)x\|_Y + \|K_0 x\|_{\mathbb{R}^n} \right)$$

$$\Longleftrightarrow \quad (1 - C\|P\|)\,\|x\|_X \leq C \left(\|(D+P)x\|_Y + \|K_0 x\|_{\mathbb{R}^n} \right)$$

$$\Rightarrow \quad \|x\|_X \leq \underbrace{\frac{C}{1 - C\|P\|}}_{C'} \left(\|(D+P)x\|_Y + \|K_0 x\|_{\mathbb{R}^n} \right)$$

By Lemma above, $D+P$ has a closed image and $\dim(\ker(D+P)) < \infty$.

By a dual argument for D $(D^*: Y^* \to X^*)$, we get a similar conclusion:

$$\exists \varepsilon > 0 \text{ s.t. } \forall p \ \|p\| < \varepsilon,$$

$$D \text{ closed image} + \dim(\operatorname{coker} D) < \infty \Rightarrow D+p \text{ closed image} + \dim(\operatorname{coker} D) < \infty.$$

<u>Prop</u>. If $D: X \to Y$ is a Fredholm operator, then $\exists \varepsilon > 0$ s.t. $\forall p: X \to Y$ with $\|p\| < \varepsilon$, we have $D+p$ is also Fredholm.

Exercise: in Prop, one can also prove $\operatorname{ind}(D+p) = \operatorname{ind}(D)$.

<u>Rmk</u> (easy to verify):

- $D: X \to Y$, $T: Y \to Z$ Fredholm $\Rightarrow \operatorname{ind}(TD) = \operatorname{ind}(D) + \operatorname{ind}(T)$.
- $D: X \to Y$ Fredholm $\Rightarrow \operatorname{ind}(D^*) = -\operatorname{ind}(D)$.

Set

$$\operatorname{Fred}(X, Y) := \{ D: X \to Y \text{ bounded linear operator} \mid D \text{ is Fredholm} \}$$

One can associate a top

<u>Cor</u> If $\gamma: [0,1] \to \operatorname{Fred}(X, Y)$ is a continuous path of Fredholm operator, then $\operatorname{ind}(\gamma(0)) = \operatorname{ind}(\gamma(1))$.

<u>2. Fredholm maps</u>

$f: X \to Y$ a smooth map and it's called a Fredholm map

X, Y: Banach mfd (locally modelled by Banach space) if $\forall x \in X$, $df(x) : T_x X \to T_{f(x)} Y$ is a Fredholm operator.

Important! Assume X is path connected, then $x \overset{\text{path}}{\sim} x' \Rightarrow df(x) \overset{\text{path}}{\sim} df(x')$

$\Longrightarrow$ $\mathrm{ind}(df(x))$ is independent of $x \in X$

$\Rightarrow$ denote $\mathrm{ind}(f)$ $\big(= \mathrm{ind}(df(x))\ \forall x \in X\big)$.

Recall that in smooth mfd, $f : M \to N$ and concept "regular value".
the same concept in Banach mfd setting.

- $y \in Y$ is called a regular value if $\forall x \in f^{-1}(y)$ we have $df(x)$ is surjective $\underline{\text{AND}}$ $df(x)$ has a right inverse.

In finite dim'l setting. surjectivity $\Rightarrow \exists$ right inverse
In ∞-dim'l setting, $\exists$ right inverse iff $\ker(df(x))$ has a complement in $T_x X$.

In particular, when f is Fredholm, then $y \in Y$ is a regular value of $f : X \to Y$ if $df(x)$ is surjective (enough!)

$$\Rightarrow \qquad \mathrm{ind}(f) = \dim(\ker df(x)) \text{ for } \forall x \in f^{-1}(y).$$

Thm (Implicit function Thm).
$f : X \to Y$ (Banach space), $y \in Y$ a regular value of f, then $M := f^{-1}(y) (\subset X)$ is a Banach mfd and $T_x M = \ker df(x)$ $\forall x \in M$.

Moreover, if f is Fredholm ~~then~~ and M is path connected then $\dim M = \mathrm{ind}(f)$ (finite!).

Thm (Smale) Let X, Y separable Banach mfd, $f: X \to Y$ (smooth) Fredholm map. Then $\{$regular value $y \in Y\}$ is residual, i.e. contains a countable intersection of open and dense sets.

Back to our J-hol curve setting:

$$\begin{array}{ccl} \mathcal{E} & = & \bigcup \{u\} \times W^{k-1,p}(\Omega^{0,1}(\Sigma, u^*TM)) \\ \bar\partial_J \circlearrowleft & \downarrow & \\ \mathcal{B} & = & W^{k,p}(\Sigma, M) \end{array}$$

Prop Let $k \geq 1$ and $p > 2$, if $u \in W^{k,p}(\Sigma, M) \cap C^1(\Sigma, M)$, the linearization $D_u: W^{k,p}(u^*TM) \longrightarrow W^{k-1,p}(\Omega^{0,1}(\Sigma, u^*TM))$ is Fredholm.

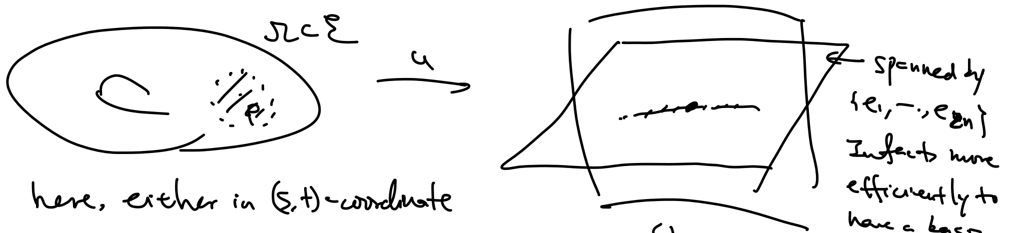
assume J is smooth

Pf (sketch) One check this locally. Recall in SFT-2, we have a local computation of D_u.

nec. to have $u \in C^1$.

$$D_u(\xi) = \bar\partial_J \xi + \frac{1}{2}\Big((\partial_\xi J)(u)\,\partial_t u\, ds - (\partial_\xi J)(u)\cdot \partial_s u\, dt\Big) \qquad (*)$$

In terms of a local basis $\{e_1, \dots e_{2n}\}$ along u (small NBH of a pt in Σ)



here, either in (s,t)-coordinate or in $(z, \bar z)$-coordinate

Ω open subset of $\mathbb{C}^n$

each $\xi \in W^{k,p}(u^*TM|_\Omega)$ is written by

$$\xi = f_1 e_1 + \cdots + f_{2n} e_{2n} \qquad f_i : \Omega \to \mathbb{R}^{2n} \simeq \mathbb{C}^n$$

and $f_i \in W^{k,p}(\Omega : \mathbb{C}^n)$

In (*), only the first term $\bar{\partial}_J \xi$ involves the derivatives of f_i.

Also, in terms of $(z, \bar{z})$-coordinate, we have $\bar{\partial}_J \xi = \partial_{\bar{z}} \xi$

$$\Longrightarrow \quad \underset{W^{k-1,p}}{\overset{\in}{D_u \xi}} = \partial_{\bar{z}} \xi + A(z)\xi \quad \text{where } A : W^{k-1,p}(\Omega, \mathrm{End}(\mathbb{C}^n))$$

(as a matrix, each component is a fcn in $W^{k-1,p}(\Omega, \mathbb{C}^n)$)

$$\Longleftrightarrow \quad \partial_{\bar{z}} \xi = D_u \xi - A(z)\xi$$

$$\xRightarrow[\text{in SFT-3}]{\text{Thm (last)}} \quad \|\xi\|_{k,p} \leq C\left(\|\xi\|_p + \|D_u \xi - A(z)\xi\|_{k-1,p}\right)$$

$$\leq C\left(\|\xi\|_{k-1,p} + \|D_u \xi\|_{k-1,p} + \|A(z)\xi\|_{k-1,p}\right)$$

$$\begin{pmatrix} * & \cdots & * \\ & & \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} \Sigma * \cdot f_i \\ \vdots \end{pmatrix}$$

- $W^{k,p} \overset{\text{emb.}}{\hookrightarrow} W^{k-1,p}$ + $W^{k-1,p}$ is closed under multiplication, we get

$$\|A(z)\xi\|_{k-1,p} \leq C' \|\xi\|_{k-1,p}$$

$$\Longrightarrow \quad \|\xi\|_{k,p} \leq C'' \left(\|D_u \xi\|_{k-1,p} + \|\xi\|_{k-1,p}\right)$$

- $W^{k,p} \overset{\text{cpt}}{\hookrightarrow} W^{k-1,p}$ + useful lemma, we get D_u is semi-Fredholm.

- By a dual argument (consider D_u^*), one conclude that D_u is Fredholm.
  Exe.

Therefore, in order to prove $\mathcal{M}_J = \left\{ \substack{\text{moduli space} \\ \text{of } J\text{-hol curve}} \right\} = \underbrace{\bar\partial_J^{-1}(0)}_{\wedge}$ is a mfd, we

just need to show $\forall u \in \bar\partial_J^{-1}(0)$, D_u is surjective. ← we actually arrive at this pt by an informal argument near the end of SFT-2.

3. Remarks

Rmk If so, then

$$\dim \mathcal{M}_J = \dim \ker(D_u) \quad \forall u \in \bar\partial_J^{-1}(0)$$

One ambiguity: $\bar\partial_J^{-1}(0)$ may not be (path) connected!

$$\bar\partial_J^{-1}(0) \underset{\substack{\uparrow \\ \text{one way to decompose}}}{=} \bar\partial_{J,A}^{-1}(0) \sqcup \bar\partial_{J,B}^{-1}(0) \cup \cdots$$

A, B htp class of $\mathrm{im}(u)$ in $H_2(M; \mathbb{R})$ (possibly with asymptotic ends/or boundary conds).

$\Rightarrow$ one consider $\mathcal{M}_{A,J}$ (i.e. fix J but with further top constraints)

and $\dim \mathcal{M}_{A,J}$ should depend on class A (later lectures).

Moreover, in each connected component, say $\bar\partial_{J,A}^{-1}(0)$,

$$\{u_t\}_{t\in[0,1]} \rightsquigarrow D_{u_t} \Rightarrow \mathrm{ind}(D_u) \text{ is ind of the path}$$

$\Rightarrow$ a benefit to calculate $\mathrm{ind}(D_u)$ from some "special" $u \in \bar\partial_{J,A}^{-1}(0)$.

Rmk (a delicate point). Does $\mathrm{ind}(D_u)$ (or more precisely $\ker(D_u)$ or $\mathrm{coker}(D_u)$) depends on the regularity degree k? NO

$\underbrace{J \text{ is smooth}}_{\substack{\text{assumption} \\ \text{and } u \text{ is } J\text{-hol.}}} \xRightarrow[\text{last time}]{\text{then}} u \in W^{k,p}(\Sigma, M) \cap C^1$ is in fact smooth $\Rightarrow$ then above applies for any $k \geq 1$.

$\Rightarrow \xi \in \ker(D_u) \subset W^{k,p}$ is in fact smooth.