- $\varphi * f \in C^k(\mathbb{C})$ b/c $*$ preserve regularity.

- If $f \in C^1_c(\mathbb{C})$, then

$$\partial_z(\varphi * f) = \varphi * \partial_z f \quad \text{and} \quad \partial_{\bar z}(\varphi * f) = \varphi * \partial_{\bar z} f$$

- Take $f = \partial_{\bar z} u$ for some $u \in C^1_0(\mathbb{C})$, then

$$\int_{\mathbb{C}} \frac{\partial_{\bar z} u}{z - z_0} dz \wedge d\bar z = \underbrace{\int_{\mathbb{C} \setminus B_\varepsilon(z_0)} \frac{\partial_{\bar z} u}{z - z_0} dz \wedge d\bar z}_{A} + \underbrace{\int_{B_\varepsilon(z_0)} \frac{\partial_{\bar z} u}{z - z_0} dz \wedge d\bar z}_{B}$$

For B, $|\partial_{\bar z} u|$ is bounded and $\int_{B_\varepsilon(z_0)} \frac{1}{|z - z_0|} dz \wedge d\bar z \overset{\text{change to polar coordinate}}{=} O(\varepsilon) \to 0$ as $\varepsilon \to 0$.

$\Rightarrow$ focus on A (and $\varepsilon \to 0$ eventually)

$$\int_{\mathbb{C} \setminus B_\varepsilon(z_0)} \frac{\partial_{\bar z} u}{z - z_0} dz \wedge d\bar z = -\int_{\mathbb{C} \setminus B_\varepsilon(z_0)} \overset{\underset{\shortparallel}{\lrcorner\, dz + \lrcorner\, d\bar z}}{d} \left(\frac{u(z)}{z - z_0} dz \right)$$

$$\overset{\text{orientation changes}}{=} + \int_{\partial B_\varepsilon(z_0)} \frac{u(z)}{z - z_0} dz$$

change variable
$z = z_0 + \varepsilon e^{\sqrt{-1}\theta}$
$dz = \varepsilon e^{\sqrt{-1}\theta} \cdot \sqrt{-1} \cdot d\theta$

$$= \int_0^{2\pi} \frac{u(z_0 + \varepsilon e^{i\theta})}{\varepsilon e^{\sqrt{-1}\theta}} \varepsilon \cdot e^{\sqrt{-1}\theta} \cdot \sqrt{-1}\, d\theta$$

$$= \int_0^{2\pi} u(z_0 + \varepsilon e^{i\theta}) \cdot \sqrt{-1}\, d\theta$$

$$\Rightarrow (\varphi * (\partial_{\bar z} u))(z_0) = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_0^{2\pi} u(z_0 + \varepsilon e^{i\theta})\, d\theta = u(z_0)$$

average value of u around z_0

$$\Rightarrow \quad \varphi * (\partial_{\bar z} u) = u.$$

$$\Rightarrow \quad \partial_{\bar z} (\varphi * u) = u$$

this means for given $f \in C_0^1(\mathbb{C})$, we can find a solution of equ $\partial_{\bar z} u = f$ where $u = \varphi * f \in C^1(\mathbb{C})$.

<u>Exercise</u> For $\Omega \subset \mathbb{C}$ be a bounded domain, $p < +\infty$, for any $f \in L^p(\Omega)$, we have f is a weak $\partial_{\bar z}$-derivative of $\varphi * f$ in the sense that $\forall \varphi \in C_0^\infty(\mathbb{C})$, $\int_\Omega (\varphi * f)\, \partial_{\bar z} g \, dA = -\int_\Omega f g \, dA$.

Now, introduce an operator $T : C_0^\infty(\mathbb{C}) \to C^\infty(\mathbb{C})$ by

$$T(\cdot) := (\partial_z \circ \varphi *)(-)$$

Then $T(\partial_{\bar z} u) = \partial_z \circ (\varphi * \partial_{\bar z} u) = \partial_z (\partial_{\bar z} (\varphi * u)) = \partial_z u.$

i.e. T switch the differential part.

<u>Rmk</u> One can extend all discussion above for $\varphi *$ and T to $C^\infty(\mathbb{C}, \mathbb{C}^n)$.

Back to the non-linear Cauchy-Riem equ $(*)$, it rewrites as

$$\overset{\partial_{\bar z} v + ((q \circ w) T)(\partial_{\bar z} v)}{\overset{\shortparallel}{\underbrace{(1 + (q \circ w) T)}_{\text{assuming it is invertible (in some sense, later)}} (\partial_{\bar z} v)}} = (\partial_{\bar z} \chi) \cdot w + (\partial_z \chi)(q \circ w) w.$$

$\Rightarrow$ ((locally) J-hol curve (solving a PDE)) is basically solving $\underline{\underline{\partial_{\bar z} v = f}}$ for some f, v defined on $\mathbb{C}$.

6. Regularity ← important but difficult

Meaning: given $f\ (= \partial_{\bar z} v) \in W^{k,p}$, how about any solution v?

Thm (Calderón-Zygmund) For $1<p<\infty$, $\exists\, C_p^{>0}$ s.t. $\forall f \in C_0^\infty(\mathbb{C})$, we have the estimate

$$\|Tf\|_p \le C_p \|f\|_p$$

So T extends to a bounded linear operator on $L^p(\mathbb{C})$. ← simply by approximation b/c $C_0^\infty(\mathbb{C})$ is dense in $L^p(\mathbb{C})$.

Rmk When $p=2$, one can manually check that $\|Tf\|_p = \|f\|_p$.

Here is an example how we play with Thm above.

Ex Suppose $\partial_{\bar z} v = f$ in a weak derivative sense where $v, f \in L^p(\mathbb{C})$ and cptly supported. Consider

$$v_n = \underset{\text{mollifer supp in } B_{\frac{1}{n}}(0)}{\eta_{\frac{1}{n}}} * v \xrightarrow{L^p} v \quad (\text{see pages above})$$

- $\partial_{\bar z} v_n = \eta_{\frac{1}{n}} * \partial_{\bar z} v \overset{\text{weak}}{=} \eta_{\frac{1}{n}} * f \xrightarrow{L^p} f$

$$\begin{aligned}
\Rightarrow \quad \|v_n - v_m\|_{1,p} &\le \|v_n - v_m\|_p + \|\partial_{\bar z}(v_n - v_m)\|_p + \|\partial_z (v_n - v_m)\|_p \\
&= \|v_n - v_m\|_p + \|\partial_{\bar z}(v_n - v_m)\|_p + \|T\partial_{\bar z}(v_n - v_m)\|_p \\
&\le \|v_n - v_m\|_p + (1 + C_p)\|\partial_{\bar z} v_n - \partial_{\bar z} v_m\|_p \longrightarrow 0
\end{aligned}$$

$\Rightarrow \{v_n\}$ is a Cauchy sequence in $W_0^{1,p}(\mathbb{C})$ $\leftarrow$ Here is a reason why we need put everything in a Banach space.

$\Rightarrow v \in W_0^{1,p}(\mathbb{C})$

Moreover,

$$\|v\|_{1,p} \leq \|v\|_p + \|\partial_{\bar z} v\|_p + \|\partial_z v\|_p$$
$$= \|v\|_p + \|f\|_p + \|Tf\|_p \leq \|v\|_p + (1+C_p)\|f\|_p. \quad //$$

Then by induction, one can get the following conclusion:

<u>Prop</u> $\partial_{\bar z} v = f$ for $v \in L_0^p(\mathbb{C})$, $f \in W_0^{k,p}(\mathbb{C})$, then $v \in W_0^{k+1,p}(\mathbb{C})$ and

$$\|v\|_{k+1,p} \leq C(\|v\|_p + \|f\|_{k,p}) \quad \text{for } C = C(k,p).$$

<u>Rmk</u> By using "multiply by χ" (cut off) technique, one can prove

$\partial_{\bar z} v = f$ for $v \in L^p(\Omega)$ and $f \in W_0^{k,p}(\Omega)$, then for all $\Omega' \subset\subset \Omega$ (cpt) we have $v \in W^{k+1,p}(\Omega')$ with the same estimate on $W^{k+1,p}$-norm.

Finally, we have to deal with the invertibility of $1 + (q \circ w)T$ near the end of page 16. Assumption: $w \in W_0^{k,p}(\mathbb{C})$ ($\Leftrightarrow u \in W^{k,p}(\mathbb{C})$) $J \in C^{k+1}(\mathbb{C}^n, \mathrm{End}(\mathbb{C}^n))$

<u>Easy case</u>: when $\|w\|_{k,p} < \varepsilon$.

- Since T commutes with D^α, T extends to $W^{k,p}(\mathbb{C})$.
- $q \in C^{k+1}(\mathbb{C}^n, \mathrm{End}(\mathbb{C}^n)) \Longrightarrow q \circ w \in W_0^{k,p}(\mathbb{C})$ by prop on page 11.

$q(z)$ = matrix and each component is C^{k+1}

- $(q\circ w)\cdot T(f) \in W^{k,p}(\mathbb{C})$ by prop on page 9.
  $T(f) \in W^{k,p}(\mathbb{C})$

and $\|(q\circ w)\cdot Tf\|_{k,p} \leq C\|q\circ w\|_{k,p}\cdot\|Tf\|_{k,p}$

$\leq \frac{1}{2}\|f\|_{k,p}$ when $\|w\|_{k,p} < \varepsilon$. for a sufficiently small ε.

Then one can define

$$\left(1+(q\circ w)\cdot T\right)^{-1} = 1-(q\circ w)\cdot T + (q\circ w\cdot T)^2 + \cdots$$

This convergence since the operator norm of $(q\circ w)\cdot T$ is <1.

<u>Hard case</u>: without the condition $\|w\|_{k,p} < \varepsilon$.
recall that in reality $w = \eta\cdot u$ + extension.

Renormalization trick: modulo the cut-off fcn χ, let's assume $u: D \to \mathbb{C}^n$ and $u \in W^{k,p}(D)$. Then consider

$$u_m(z) := u\left(\frac{z}{m}\right) \quad \text{for } z\in D \text{ and } k\in\mathbb{N}_{\geq 1}.$$

Then $\partial_{\bar z}u_m + (q\circ u_m)(z)\,\partial_z u_m = \frac{1}{m}\partial_{\bar z}u\left(\frac{z}{m}\right) + q\left(u\left(\frac{z}{m}\right)\right)\cdot\frac{1}{m}\partial_z u\left(\frac{z}{m}\right)$

recall the middle of page 13

$= 0$ (if u is J-hol).

Key Claim: $\|u_m\|_{k,p} \to 0$ as $m \to 0$.

$\Rightarrow$ $\exists\, m_0$ sufficiently large s.t. $\|u_m\|_{k,p} < \varepsilon$. (so the modified w_m satisfies $\|w_m\|_{k,p} < \varepsilon$. and EASY case above applies. $\leftarrow$ the resulting supp changes from D to $\frac{1}{m_0}D$.

Pf of the claim (extra hypothesis needed!)
added along the argument

For $|\alpha| \geq 1$,

$$\int_{\mathbb{D}} |D^\alpha u_m(z)|^p \, dV(z) = \int_{\mathbb{D}} |D^\alpha u(\tfrac{z}{m})|^p \, dV(z)$$

$$= \int_{\mathbb{D}} |m^{-|\alpha|} D^\alpha u(\tfrac{z}{m})|^p \, dV(z)$$

$$\underset{dV(z) = \frac{1}{m^2} dV(x)}{\overset{x = \frac{z}{m}}{=}} \int_{\frac{1}{m}\mathbb{D}} |m^{-|\alpha|} D^\alpha u(x)|^p \cdot m^2 \, dV(x)$$

$$= \int_{\frac{1}{m}\mathbb{D}} m^{2-p|\alpha|} |D^\alpha u(x)|^p \, dV(x) \leq m^{2-p|\alpha|} \|u\|_{k,p}$$

(over $\frac{1}{m}\mathbb{D}$).

extra hypo 1

Assume $p > 2$, then $2 - p|\alpha| < 0$, so $D^\alpha u_m \xrightarrow{L^p} 0$.

For $\alpha = 0$, by Sobolev emb (Thm on page 7) where $p > 2 \; ({}^n)$ and assume

extra hypo 2.

$k \geq 1$, we know

$$W^{k,p}(\mathbb{D}) \overset{\text{cont.}}{\hookrightarrow} C^{k-1, \gamma = 1 - \frac{2}{p}}(\mathbb{D})$$

we can/have assumed $u(0) = 0$

$$\Rightarrow u \; (\in W^{k,p}(\mathbb{D})) \text{ satisfies } \frac{|u(\frac{z}{m}) - u(0)|}{|\frac{z}{m} - 0|^{1-\frac{2}{p}}} \leq C \cdot \|u\|_{k,p}$$

$$\Rightarrow |u_m(z)| = |u(\tfrac{z}{m})| \leq |\tfrac{z}{m}|^{1-\frac{2}{p}} \cdot C \cdot \|u\|_{k,p}$$

$$\leq \left|\frac{1}{m}\right|^{\overbrace{1-\frac{2}{p}}^{>0}} \cdot C''(u) \quad b/c \; |z| \leq \text{radius of } \mathbb{D}.$$

$\Rightarrow$ $u_m \to 0$ uniformly and then $u_m \in L^p(\mathbb{D})$

Therefore, $\|u_m\|_{k,p} \to 0$. $\square$

To summarize what we have done for this lecture, we get the following regularity result.

<u>Thm</u> (Thm 2.24 (i) in [Wen]) $u: (\mathbb{D}, j) \to (\mathbb{C}^n, J)$ J-hol, where J is C^k and $k \geq 1$,

if $u \in W^{1,p}(\mathbb{D}, \mathbb{C}^n)$ with $p > 2$, then $\exists\, \mathbb{D}' \subset \mathbb{D}$ s.t. $u \in W^{k,p}(\mathbb{D}', \mathbb{C}^n)$

In particular, if J is smooth, then $u \in C^\infty(\mathbb{D}'', \mathbb{C}^n)$ $\leftarrow$ b/c $p > 2 = n$ and $W^{k,p} \hookrightarrow C^{k-1}$!
for some open disk $\mathbb{D}'' \subset \mathbb{D}$.

Note that this Thm forms the "local statement" that one can upgrade to a statement for $W^{k,p}(\Sigma, M)$ (see Next Lect - SFT 4).