One can keeping applying result above.

**Ex** Assume $k<\frac{n}{p}$, then

$$W^{k,p}(\Omega) \hookrightarrow W^{k-1,p^*}(\Omega) \hookrightarrow W^{k-2,p^{**}}(\Omega) \hookrightarrow \cdots \hookrightarrow W^{0,\overset{q}{\overset{\shortparallel}{p^{\overbrace{*\cdots*}^{k}}}}}(\Omega)$$

← by comments above this can be any $q\in[1,p^*]$

$$\frac{1}{p}-\frac{1}{p^*}=\frac{1}{n} \qquad \frac{1}{p^*}-\frac{1}{p^{**}}=\frac{1}{n} \qquad \frac{1}{p}-\frac{1}{q}=\frac{k}{n}$$

$$\left(\Leftrightarrow \frac{1}{p}-\frac{1}{p^{**}}=\frac{2}{n}\right)$$

$$\Rightarrow W^{k,p}(\Omega)\hookrightarrow L^q(\Omega) \text{ when } \frac{1}{p}-\frac{1}{q}=\frac{k}{n}.$$

in facts holds for any $p\le q\le p^*$

**Ex** Assume $k>\frac{n}{p}$. We will discuss in two cases.

- If $\frac{n}{p}\notin\mathbb{N}$, then take $l=\lfloor\frac{n}{p}\rfloor$ (so $l<\lfloor\frac{n}{p}\rfloor<l+1$, and $k\ge l+1$)

$$W^{k,p}(\Omega)\hookrightarrow W^{k-l,q}(\Omega)\hookrightarrow C^{k-l-1,\frac{n}{q}}(\Omega)$$

$$\frac{1}{p}-\frac{1}{q}=\frac{l}{n}$$

Here $\frac{n}{q}=\frac{n}{p}-l$, so $C^{k-l-1,\frac{n}{q}}=C^{k-\lfloor\frac{n}{p}\rfloor-1,\ \frac{n}{p}-\lfloor\frac{n}{p}\rfloor}$.

The hard part is that the standard argument will arrive at the "borderline" case for Sobolev emb when $p=n$.

- If $\frac{n}{p}\in\mathbb{N}$, (Exercise) $W^{k,p}(\Omega)\hookrightarrow C^{k-\lfloor\frac{n}{p}\rfloor-1,\beta}(\Omega)$ for any $\beta\in(0,1)$.

**Cor.** (Prop 2.4 in [Wen])

(1) If $k>\frac{n}{p}$, then for any integer $d\ge 0$, $W^{k+d,p}(\Omega)\hookrightarrow C^d(\bar\Omega)$

b/c $W^{k+d,p}(\Omega)\hookrightarrow C^{d+\underbrace{(k-\lfloor\frac{n}{p}\rfloor-1)}_{\ge 0},\beta}(\bar\Omega)\subset C^{d+(k-\lfloor\frac{n}{p}\rfloor-1)}(\bar\Omega)\subset C^d(\bar\Omega)$.

(2) $k\ge m$ in $\mathbb{N}$, $p\le q$ and $k-\frac{n}{p}\overset{>}{\geq} m-\frac{n}{q}$, then

$$W^{k,p}(\Omega)\hookrightarrow W^{m,q}(\Omega).$$

I can only derive this conclusion under $>$ $k-\frac{n}{p}$

(BUT it seems many book also list this result without this cond.)

b/c $W^{k,p}(\Omega) \hookrightarrow W^{k-(k-m),p^*} = W^{m,p^*}$

$$\frac{1}{p} - \frac{1}{p^*} = \frac{k-m}{n} \quad \left(\Leftrightarrow k - \frac{n}{p} = m - \frac{n}{p^*}\right)$$

Then the same argument works for any $q \in [p, p^*]$ when $k - \frac{n}{p} \geq m - \frac{n}{q}$.

Finally, recall an operator $F: \underbrace{X \to Y}_{\text{Banach space}}$ is called cpt if any bounded seq $\{x_n\}$ in $X$, its image $\{F(x_n)\}_n$ admits a converging subsequence in $Y$.

<u>Thm</u> (Rellich-Kondrachov compactness)

Embeddings (as inclusions) in Cor above are compact, when $\Omega$ is bounded. (and in the second case $\geq$ is strict).

<u>Ex</u> When $\Omega$ is bounded, $W^{k,p}(\Omega) \xhookrightarrow{\text{compact}} W^{k-1,p}(\Omega)$

Here, $m = k-1$, $q = p$ (so $k - \frac{n}{p} \geq (k-1) - \frac{n}{p} = m - \frac{n}{q}$).

## 4. Corollaries

<u>Prop</u> $\Omega \subset \mathbb{R}^n$, $p>n, k \geq 1$, $W^{k,p}(\Omega)$ is an algebra under multiplication.

i.e. $f, g \in W^{k,p}(\Omega)$, $fg \in W^{k,p}(\Omega)$. Moreover, $\exists$ a uniform $C>0$ s.t.

$$\|fg\|_{k,p} \leq C \|f\|_{k,p} \|g\|_{k,p}.$$

<u>pf.</u>

If $f, g \in C_c^\infty(\Omega)$

Morrey's inequality $\underset{\text{apply to } D^\alpha f \in W^{1,p}}{\Longrightarrow}$ $\forall \alpha$ with $|\alpha| \leq k-1$,

$$\max |D^\alpha f| \leq A \|D^\alpha f\|_{1,p} \leq A \|f\|_{k,p}.$$

Similarly, $\max |D^\alpha g| \leq A \|g\|_{k,p}$.

By Lebniz rule, for $\alpha$ s.t $|\alpha| \leq k$, we have

$$D^\alpha (fg) = \sum_{\beta+\gamma=\alpha} (D^\beta f)(D^\gamma g)$$

so $|\beta| \leq k-1$ or $|\gamma| \leq k-1$.

$\Rightarrow$ If $|\beta| \leq k-1$, then $\|(D^\beta f)(D^\gamma g)\|_p \leq \max |D^\beta f| \, \|D^\gamma g\|_p$

$$\lesssim A \|f\|_{k,p} \|g\|_{k,p}$$

Similarly, if $|\gamma| \leq k-1$, then $\|(D^\beta f)(D^\gamma g)\|_p \leq A \|f\|_{k,p} \|g\|_{k,p}$.

$\Rightarrow \forall \alpha$ with $|\alpha| \leq k$, $\|D^\alpha (fg)\|_p \leq A \sum_{\beta+\gamma=\alpha} \|f\|_{k,p} \|g\|_{k,p}$

$$= C \|f\|_{k,p} \|g\|_{k,p}$$

Now, approximate $f_n \to f$ and $g_n \to g$ where $f_n, g_n \in C_0^\infty(\Omega)$.

Then

$$\|f_m g_m - f_n g_n\|_{k,p} = \| f_m (g_m - g_n) + (f_m - f_n) g_n \|_{k,p}$$

$$\underset{\text{by discussion above}}{\leq} C \|f_m\|_{k,p} \|g_m - g_n\|_{k,p} + C \|g_n\|_{k,p} \|f_n - f_m\|_{k,p}$$

$\Rightarrow \{f_n g_n\}_n$ is a Cauchy sequence in $W^{k,p}(\Omega)$.

$\Rightarrow$ its limit $fg \in W^{k,p}(\Omega)$ and $\|fg\|_{k,p} \leq C \|f\|_{k,p} \|g\|_{k,p}$. $\square$.

Rmk (Exercise) $\Omega \subset \mathbb{R}^n$, $p, q \in [1,\infty)$, $k, m \in \mathbb{N}$ s.t. $k \geq m$, $kp > n$ and $k - \frac{n}{p} \geq m - \frac{n}{q}$, $\exists$ a continuous embedding via product

$$W^{k,p}(\Omega) \times W^{m,q} \hookrightarrow W^{m,q}.$$

(cf. $f \in C^k$, $g \in C^m \Rightarrow fg \in C^m$).

**Prop** $\Omega \subset \mathbb{R}^n$ bounded, $p > n$, $k \geq 0$ $g: \mathbb{R}^m \to \mathbb{R}$ cpt supp $C^{k+1}$-fcn then $\forall$ $u \in W^{k,p}(\Omega; \underbrace{\mathbb{R}^m}_{\text{target}})$ and cpt supp, then $g \circ u \in W^{k,p}(\Omega)$.

**Inv.**
$k=0$ case: $g$ cpt supp $C^1$-fcn $\Rightarrow$ $|g(\underbrace{x}_{\mathbb{R}^m}) - g(0)| \leq \max \|\nabla g\| |x|$

$$\Rightarrow |(g \circ u)(z)| \leq \max \|\nabla g\| |u(z)| + |g(0)|$$

This holds for any $z \in \Omega$, so

$$\|g \circ u\|_p \leq \max \|\nabla g\| \underbrace{\|u\|_p}_{<\infty \text{ b/c } u \in W^{0,p}(\Omega) = L^p(\Omega)} + |g(0)| \, vol(\Omega)^{1/p} \Rightarrow g \circ u \in W^{0,p}(\Omega).$$

$k=1$ case: $g$ cpt supp $C^2$-fcn
$u \in W^{1,p}(\Omega)$. in this setting one can show

$$\frac{\partial (g \circ u)}{\partial x_i} = \sum_{r=1}^{m} \left( \frac{\partial g}{\partial x_r} \circ u \right) \cdot \underbrace{\frac{\partial u}{\partial x_i}}_{\text{in the sense of weak derivative.}} \quad (*)$$

Along the proof, one shows that $\frac{\partial u}{\partial x_i} \in L^p(\Omega)$.

Then we confirm this case from the discussion for $k=0$. b/c $\frac{\partial (g \circ u)}{\partial x_i} \in L^p$ and $g \circ u \in L^p$

**Pf** By induction (where the inductive hypothesis is that for the conclusion holds for $k-1$: $C_0^k(\mathbb{R}^m) \times W_0^{k-1,p}(\Omega) \to W^{k-1,p}(\Omega)$

Again, using $(*)$, for $g \in C_0^{k+1}(\mathbb{R}^m)$ and $u \in W_0^{k,p}(\Omega)$,

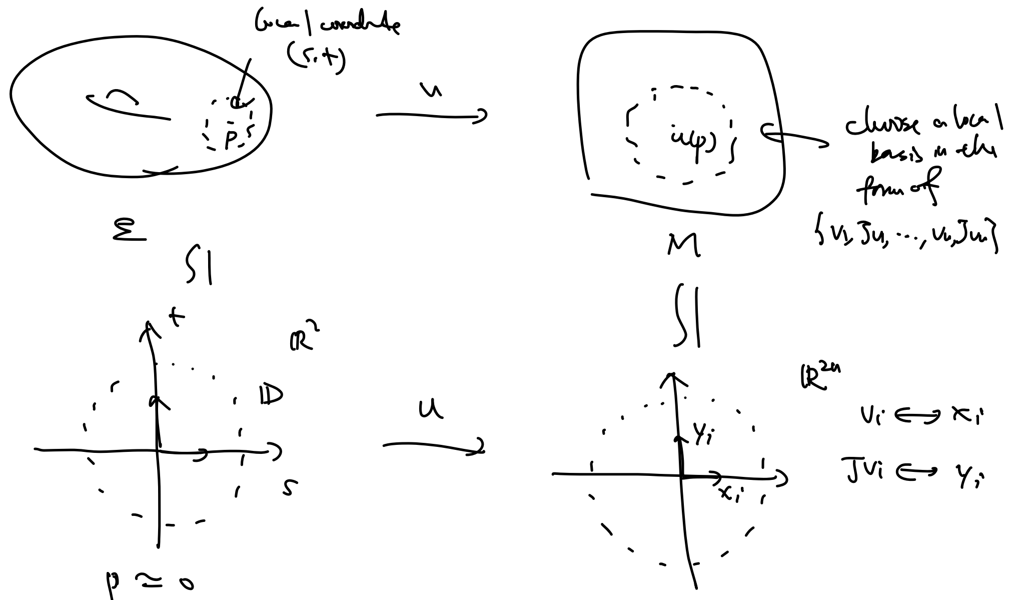
$$\underbrace{\left( \frac{\partial g}{\partial x_r} \circ u \right)}_{\text{inductive hypothesis} \to\, \in\, W^{k-1,p}(\Omega)} \cdot \underbrace{\frac{\partial u}{\partial x_i}}_{\text{hypothesis} \to\, \in\, W_0^{k-1,p}(\Omega)} \underset{\text{algebra str.}}{\Rightarrow} \frac{\partial (g \circ u)}{\partial x_i} \in W^{k-1,p}(\Omega)$$

$\square$

## 5. Solutions of Cauchy-Riemann equations

$u : (\Sigma, j) \to (M, J)$ satisfying $J \cdot u_* = u_* \cdot j$.

In local coordinate $(s,t)$ on $\Sigma$, $\frac{\partial u}{\partial s} + J(u(s,t)) \frac{\partial u}{\partial t} = 0$.




$u(p) \simeq 0$ and $J(u(p)) \simeq J_0$ standard cpx str on $\mathbb{C}^n$.

On the level of Euclidean spaces,

$$u : \mathbb{D} \to \mathbb{R}^{2n} (\simeq \mathbb{C}^n) \qquad \frac{\partial u}{\partial s} + J(u(z)) \frac{\partial u}{\partial t} = 0$$

where $J(u(0)) = J(0) = J_0$ (and $J : \mathbb{R}^2$ (or NBH near 0) $\to \mathrm{End}(\mathbb{C}^n)$ s.t. $J^2 = -\mathbb{1}$ for all $z \in$ NBH).

To simplify the notation, one can also introduce $z = s + \sqrt{-1}\, t$ and $\bar{z} = s - \sqrt{-1}\, t$ for $\mathbb{C}$ (accordingly,

$$\partial_z := \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial s} - \sqrt{-1}\frac{\partial}{\partial t}\right) \quad \text{and} \quad \partial_{\bar{z}} := \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial s} + \sqrt{-1}\frac{\partial}{\partial t}\right)$$

careful about this sign

$$\Rightarrow \quad \partial_z u = \frac{1}{2}\left(\frac{\partial u}{\partial s} - J_0 \frac{\partial u}{\partial t}\right) \quad \text{and} \quad \partial_{\bar z} u = \frac{1}{2}\left(\frac{\partial u}{\partial s} + J_0 \frac{\partial u}{\partial t}\right)$$

$$\begin{aligned}
\Rightarrow \quad \frac{\partial u}{\partial s} + J(u(z))\frac{\partial u}{\partial t} &= (\partial_z u + \partial_{\bar z} u) + J(u(z)) \cdot J_0^{-1} (\partial_{\bar z} u - \partial_z u) \\
&= (\partial_z u + \partial_{\bar z} u) + J(u(z)) \cdot J_0 \, (\partial_z u - \partial_{\bar z} u) \\
&= \big(I - J(u(z)) J_0\big) \partial_{\bar z} u + \big(I + J(u(z)) \cdot J_0\big) \partial_z u
\end{aligned}$$

In particular, at $z = 0$ $(\simeq p)$

$$I - J(\overset{u(0)=0}{0}) \cdot J_0 = I - J_0^2 = \overset{\text{invertible}}{2I} \quad \text{and} \quad I + J(0) \cdot J_0 = 0.$$

So in a NBH of 0, $I - J(u(z)) \cdot J_0$ is invertable.

so $q(\overset{\mathbb{C}^n}{z}) = (I - J(z) \cdot J_0)^{-1} \cdot (I + J(z) \cdot J_0)$

$$\Rightarrow \quad J\text{-hol} \overset{\text{locally}}{\Longleftrightarrow} \partial_{\bar z} u + \underbrace{\big(I - J(u(z)) \cdot J_0\big)^{-1} \cdot \big(I + J(u(z)) \cdot J_0\big)}_{(q \circ u)(z) \text{ with } (q \circ u)(0) = 0.} \partial_z u = 0.$$

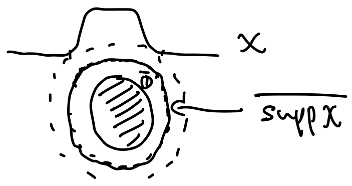
This can be regarded as a nonlinear perturbation of $\partial_{\bar z} u = 0$

Now, by choosing a cut-off fcn $\chi : \mathbb{D} \to [0,1]$ cpt supp and $\equiv 1$ in a smaller disk $\overline{\mathbb{D}'} \subset \mathbb{D}$, then for $\chi \cdot u$, we have

$$\begin{aligned}
\partial_{\bar z}(\chi \cdot u) + (q \circ u)\partial_z(\chi \cdot u) &= \partial_{\bar z}\chi \cdot u + \underline{\chi \cdot \partial_{\bar z} u} + \underline{(q \circ u)}(\partial_z \chi \cdot u + \underline{\chi \cdot \partial_z u}) \\
\text{assume } u \text{ is } J\text{-hol} \longrightarrow &= \underset{\text{cpt supp in } \mathbb{D}}{\partial_{\bar z}\chi \cdot u} + (q \cdot u)\underset{\text{cpt supp in } \mathbb{D}.}{(\partial_z \chi \cdot u)}
\end{aligned}$$

$\Rightarrow$ extend by zero to define over $\mathbb{C}$.

Set $v := \chi \cdot u$ + extension by zero and one aims to rewrite the eqn above in terms of $v$.




Rmk A subtle pt: directly replace $u$ by $v$ in $q\cdot u$ will cause a trouble due to the possible difference between $u$ and $v$ in the region $\overline{\operatorname{supp}\chi}\setminus\overline{\mathbb{D}'}$.

Resolve: Choose another cutoff fcn $\eta : \mathbb{D} \to [0,1]$ but $\eta \equiv 1$ on $\overline{\operatorname{supp}\chi}$.
Then introduce $w := \eta \cdot u$ + extension. Then $q\cdot w = q \cdot u$ on $\overline{\operatorname{supp}\chi}$.

$$\Longrightarrow \quad \partial_{\bar z} v + (q\cdot w)\,\partial_z v = \partial_{\bar z}\chi \cdot w + (q\cdot w)\,(\partial_z \chi \cdot w)$$
$$= \partial_{\bar z}\chi \cdot w + \partial_z \chi\,(q\cdot w)\,w \qquad (*)$$

Here is an even more elegant way to simplify $(*)$.

Denote by $\varphi : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ the fcn $\varphi(z) = \dfrac{1}{\pi z}$.

one can check $\varphi$ is locally integrable

Then for $f \in C^k_0(\mathbb{C})$ for some $k$, consider

$$(\varphi * f)(z_0) = \int_{\mathbb{C}} \varphi(z_0 - z)\cdot f(z)\, d\text{Area}.$$

convolution

$$d\text{Area} = ds\wedge dt = \frac{1}{2}(dz + d\bar z)\wedge\left(\frac{-\sqrt{-1}}{2}\right)(dz - d\bar z)$$

$z = s + \sqrt{-1}\,t$
$\bar z = s - \sqrt{-1}\,t$

$$= \frac{-\sqrt{-1}}{4}\cdot(-1)\cdot 2\,(dz\wedge d\bar z) = \frac{\sqrt{-1}}{2}\, dz\wedge d\bar z.$$

$$\Longrightarrow \quad (\varphi * f)(z_0) = \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{C}} \frac{f(z)}{z - z_0}\, dz\wedge d\bar z.$$

and this integral is welldefined.