1. Basics

Recall notations

- $f : \mathbb{R}^n \to \mathbb{R}^m$ measurable $p \in [1, \infty)$,

$$\|f\|_p := \left(\int_{\mathbb{R}^n} |f|^p \right)^{1/p}$$

this integration keeps the same for $g : \mathbb{R}^n \to \mathbb{R}^m$ where $m(\{x \in \mathbb{R}^n \mid f(x) \neq g(x)\}) = 0$.

When $p = \infty$, $\|\cdot\|_{L^\infty}$ = essential sup.

integrable in L^p-sense

$$L^p(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{R}^m \mid \|f\|_{L^p} < +\infty \} / \sim$$

measure 0 equivalence

<u>Prop</u> $L^p(\mathbb{R}^n)$ is a Banach space (i.e. complete normed vector space).

<u>Rmk</u> One drawback of $f \in L^p(\mathbb{R}^n)$ is that it's hard to take derivatives.

- $C(\mathbb{R}^n; \mathbb{R}^m) = \{$ continuous fcns f on $\mathbb{R}^n \}$

  $C_0(\mathbb{R}^n; \mathbb{R}^m) = \dots$ cptly supported.

  k could be ∞.

  $C^k(\mathbb{R}^n, \mathbb{R}^m) = \{ \exists$ k-th derivatives and they are cont. $\}$

To simplify the presentation, we will consider $\mathbb{R}^m = \mathbb{R}$, and $C^k(\mathbb{R}^n, \mathbb{R})$ or $C_0^k(\mathbb{R}^n, \mathbb{R})$ will be denoted by $C^k(\mathbb{R}^n)$ or $C_0^k(\mathbb{R}^n)$.

$x_1, \dots, x_n$

For a multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, define $|\alpha| = \sum_{i=1}^n \alpha_i$, then

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

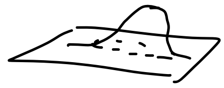
For $f \in C^k(\mathbb{R}^n)$, $D^\alpha f = \dfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ when $|\alpha| \leq k$.

• Convolution $*$

f, g two fcns on $\mathbb{R}^n$, then

$$\underset{\underset{\mathbb{R}^n}{\in}}{(f*g)(x)} := \int_{\mathbb{R}^n} f(x-y)g(y)\, d\mathrm{vol}_y \quad \leftarrow dy_1\wedge\cdots\wedge dy_n$$

$f*g = g*f$ but we will apply it in a rather non-symmetric way: take f = the standard mollifier $\eta : \mathbb{R}^n \to \mathbb{R}$

(= a radial symmetric bump fcn supported in $B_0(1)$  s.t. $\int_{\mathbb{R}^n}\eta = 1$)

For $r>0$, $\eta_r(x) := r^{-n}\eta\left(\frac{x}{r}\right)$ ← to make sure $\int_{\mathbb{R}^n}\eta_r = 1$, then for any $g \in L^p(\mathbb{R}^n)$,

→ $\eta_r * g \in C^\infty(\mathbb{R}^n)$ ← b/c $*$ preserves regularity for any input.

→ $\eta_r * g \xrightarrow{L^p} g$ as $r \to 0$

pf $\quad h \in C_0(\mathbb{R}^n) \xrightarrow{L^p} g$

$$\|\eta_r * g - g\|_p \le \underbrace{\|\eta_r*(g-h)\|_p}_{A} + \underbrace{\|\eta_r * h - h\|_p}_{B} + \underbrace{\|h-g\|_p}_{<< \varepsilon/2}$$

$$A \underset{\uparrow}{\le} \|\eta_r\|_1 \|g-h\|_p < \varepsilon/2$$

Young's inequality

B : $|(\eta_r * h)(x) - h(x)| = \left|\int_{\mathbb{R}^n} \eta_r(x-y)\,h(y) - h(x)\right|$

$= \left|\int_{\mathbb{R}^n} \eta_r(x-y)\,(h(y)-h(x))\right| \quad \leftarrow$ b/c $\int_{\mathbb{R}^n}\eta_r = 1$

$z = x-y$

$\le \int_{\mathbb{R}^n} \eta_r(z)\,|h(x-z)-h(x)| \le \int_{B^n(r)} \eta_r(z)\,|h(x-z)-h(x)| < \varepsilon/3$ □.

one should think more carefully why the estimation in B implies uniform continuity.

· weak derivative

Motivation: integration by parts $u \in C^\infty(\mathbb{R}^n)$, $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$,

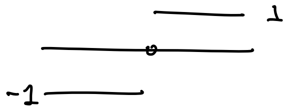
$$0 = \int_{\mathbb{R}^n} d(u\varphi) = \int_{\mathbb{R}^n} \partial u \cdot \varphi + \int_{\mathbb{R}^n} u \cdot \partial \varphi$$

b/c φ is cptly supp

$\Rightarrow$ one can ~~replace~~ formally define ∂u by a fcn f s.t. $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$, we have: (called a weak derivative of u)

$$\int_{\mathbb{R}^n} u \cdot \frac{\partial \varphi}{\partial x_i} = -\int_{\mathbb{R}^n} f \cdot \varphi$$

Ex $u = |x| : \mathbb{R} \to \mathbb{R}$. $\frac{du}{dx} \sim \sigma : \mathbb{R} \to \mathbb{R}$ by



check: $\forall \varphi \in C_0^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} |x| \frac{d\varphi}{dx} dx = \int_{-\infty}^{0} (-x) \cdot \frac{d\varphi}{dx} dx + \int_0^\infty x \cdot \frac{d\varphi}{dx} dx$$

$$\overset{\text{integration by parts}}{=} \int_{-\infty}^{0} \varphi(x) dx - \int_0^\infty \varphi(x) dx = -\int_{-\infty}^{\infty} \sigma(x)\, \varphi(x)\, dx$$

Rmk One can modify f by a zero-measure set, which can also serve as a weak derivative of u.

Prop If f is a weak derivative of u, then for $g \in C_0(\mathbb{R}^n)$, $g * f$ is a weak derivative of $g * u$.

In particular, if $g \in C_0^1(\mathbb{R}^n)$, then

$$\frac{\partial g}{\partial x_i} * u = g * f$$

$$\frac{\partial}{\partial x_i}(g * u)$$

both are weak derivatives of $g*u$, and they are both continuous.

2. Sobolev space

$k \in \mathbb{N}$ and $p \in [1, \infty)$, then

$$W^{k,p}(\mathbb{R}^n) := \left\{ u \in L^p(\mathbb{R}^n) \;\middle|\; \begin{array}{l} \forall \alpha \text{ with } |\alpha| \le k \\ \exists f_\alpha \in L^p(\mathbb{R}^n) \text{ s.t. } f_\alpha \text{ is a} \\ \text{weak derivative of } D^\alpha u \end{array} \right\}$$

Then for $u \in W^{k,p}(\mathbb{R}^n)$, define

$$\|u\|_{k,p} := \left(\|u\|_p^p + \sum_{1 \le |\alpha| \le k} \|f_\alpha\|_p^p \right)^{1/p}$$

Thm $(W^{k,p}(\mathbb{R}^n), \|\cdot\|_{k,p})$ is a Banach space.

$\Longrightarrow$ $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ $\left(\text{so } \overline{C_0^\infty(\mathbb{R}^n)}^{\|\cdot\|_{k,p}} = W^{k,p}(\mathbb{R}^n) \right)$

pf (sketch) For $u \in W^{k,p}(\mathbb{R}^n)$, one shows

$$\eta_r * u \xrightarrow{\|\cdot\|_{k,p}} u \quad \text{and} \quad \eta_r * u \in \underset{\substack{\uparrow \\ \exists \text{ weak derivatives} \\ \text{(by prop above)}}}{W^{k,p}(\mathbb{R}^n)} \cap \underset{\substack{\uparrow \\ \text{regularity}}}{C^\infty(\mathbb{R}^n)}$$

Moreover, one can approximate $\eta_r * u$ by elements in $C_0^\infty(\mathbb{R}^n)$ b/c $\eta_r * u \in W^{k,p}(\mathbb{R}^n)$ (so one can consider $\chi \cdot (\eta_r * u)$ when χ is a bump fcn with large support). ← outside this supp. L^p-norm of derivatives is small. ∎

Rmk $\left(\begin{array}{c} C_0^\infty(\mathbb{R}^n) \\ \text{or} \\ C^\infty(\mathbb{R}^n) \end{array}, |\cdot|_\infty \right)$ is not Banach.

$\frac{1}{1+x^2} = \text{limit} \notin C_0^\infty(\mathbb{R})$

But they are Fréchet space.

Def A top. v.s X is a Fréchet space if it is a Hausdorff space equipped with a family of semi-norms $|\cdot|_k$ $k \in \mathbb{N}$ s.t.

- $|x|_k = 0 \ \forall k \Rightarrow x = 0$ (non-deg)
- X is complete w.r.t to the metric $d(x,y) := \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|x-y|_k}{1+|x-y|_k}$

make it well-defined

e.g. Every Banach space is a Fréchet space ($|\cdot|_k \equiv \|\cdot\| \ \forall k$)

e.g. $C^\infty(\mathbb{R})$ can be equipped with a seq of semi-norm so that it becomes a Fréchet space.

- For $p=2$, $W^{k,2}(\mathbb{R}^n)$ admits an inner product

$$\langle u, v \rangle := \sum_{0 \le |\alpha| \le k} \int_{\mathbb{R}^n} (D^\alpha u)(D^\alpha v)$$

(this is well-defined due to Cauchy-Schwarz inequality.)

Notation: $H^k(\mathbb{R}^n) := W^{k,2}(\mathbb{R}^n)$.

- Change the input from $\mathbb{R}^n$ to an open domain $\Omega \subset \mathbb{R}^n$.

Then

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \begin{array}{l} \forall \alpha \text{ with } |\alpha| \le k \\ \exists f_\alpha \in L^p(\Omega) \text{ s.t. } f_\alpha \text{ is a} \\ \text{weak derivative of } D^\alpha u \end{array} \right\}$$

Prop If $\partial\Omega$ satisfies certain "extentible" condition, then

$$\overline{C^\infty(\bar{\Omega})}^{\|\cdot\|_{k,p}} = W^{k,p}(\Omega)$$

similarly, we can define $W_0^{k,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{k,p}}$.

For most cases, we will always assume Ω bounded and satisfies this condition. see AppA in [Evan].

(in reality, this condition is even more complicated when Ω is unbounded).

3. Embedding Thm

FACT (Morrey's inequality) $\mathbb{R}^n$, $p>n$, then $\exists\, C$ s.t. if $u \in W^{1,p}(\Omega) \cap C^\infty(\bar{\Omega})$, then $\forall x, y \in \Omega$

$$|u(x)| \leq C \|u\|_{1,p} \quad \text{and} \quad |u(x)-u(y)| \leq C \|\nabla u\|_p \, |x-y|^{1-\frac{n}{p}}$$

$$\|\nabla u\|_p = \left(\sum_{i=1}^n \int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial x_i}\right|^p \right)^{1/p}$$

using the notation below, Morrey's inequality implies

$$\|u\|_{C^{0,1-\frac{n}{p}}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}$$

Recall the γ- Hölder continuity ($\gamma \in [0,\infty)$) (stronger than uniformly continuous)

$$\exists\, C>0 \text{ and } \gamma \text{ s.t } |u(x)-u(y)| \leq C|x-y|^\gamma \quad \forall x, y$$

Rmk
- $\gamma = 0 \Rightarrow u$ is bounded
- $\gamma > 1 \Rightarrow u$ is constant
- $\gamma = 1 \Rightarrow u$ is Lipschitz

$\Rightarrow$ usually one considers $\gamma \in (0,1]$.

$$C^{k,\gamma}(\Omega) := \left\{ u \in C^k(\Omega) \;\middle|\; \begin{array}{l} |\alpha| = k,\ D^\alpha u \text{ is} \\ \gamma\text{-Hölder continuous} \end{array} \right\}$$

$$\|u\|_{C^{k,\gamma}} := \|u\|_{C^k} + \max_{|\alpha|=k} \sup_{x\neq y \in \Omega} \frac{(D^\alpha u)(x) - (D^\alpha u)(y)}{|x-y|^\gamma}$$

Prop $(C^{k,\gamma}(\Omega), \|\cdot\|_{C^{k,\gamma}})$ is a Banach space (called Hölder space)

Ex If $\gamma' < \gamma$, then $C^{k,\gamma}(\Omega) \subsetneq C^{k,\gamma'}(\Omega)$ and not dense. $\hookrightarrow$ is compact (later)

Thm (Sobolev emb for $\Omega \subset \mathbb{R}^n$, $p > n$)

$$W^{k,p}(\Omega) \xhookrightarrow[\text{cont}]{S} C^{k-1,\, 1-\frac{n}{p}}(\Omega)$$

($\Leftrightarrow$ bounded operator)

Meaning:

① $\forall\, u \in W^{k,p}(\Omega)$, $S(u) =$ a modification of u in a 0-measured set in Ω.

② $\|S(u)\|_{C^{k-1,1-\frac{n}{p}}} \le C \|u\|_{k,p}$ for a uniform constant C.

pf. • $u_m \xrightarrow{\|\cdot\|_{k,p}} u$ where $u_m \in W^{k,p}(\Omega) \cap C^\infty(\Omega)$.

$\overset{\text{Morrey part 2}}{\Longrightarrow}$ $\{D^\alpha u_m\}_m$ (for each fixed α with $|\alpha| \le k-1$) is a Cauchy sequence in $W^{1,p}$-sense.

$\Rightarrow$ $D^\alpha u_m \to V^\alpha$ and then $D^\alpha u$ exists and equals to V^α.
($u_m \to u$)

• Use Morrey part 2 to estimate $D^\alpha u$ and the correct Hölder exponent is precisely $1-\frac{n}{p}$. □

FACT (Gagliardo-Nirenberg-Sobolev) $1 \le p < n$ and p^* satisfies $\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}$
then $\exists\, c > 0$ s.t. for all $u \in C_0^\infty(\Omega)$ ($\mathbb{R}^n$), we have

$$\|u\|_{p^*} \le C \|\nabla u\|_p .$$

← this implies $\forall_{1 \le q \le p^*}$, $\|u\|_q \le C' \|\nabla u\|_p$ by Hölder inequality.

Thm (Sobolev emb for $\Omega \subset \mathbb{R}^n$, $p < n$)

$$W^{k,p} \xhookrightarrow[\text{cont.}]{S} W^{k-1,\, p^*}$$

in GNS inequality

* degree of regularity drops.

pf Similar to the one above after apply GNS inequality. □