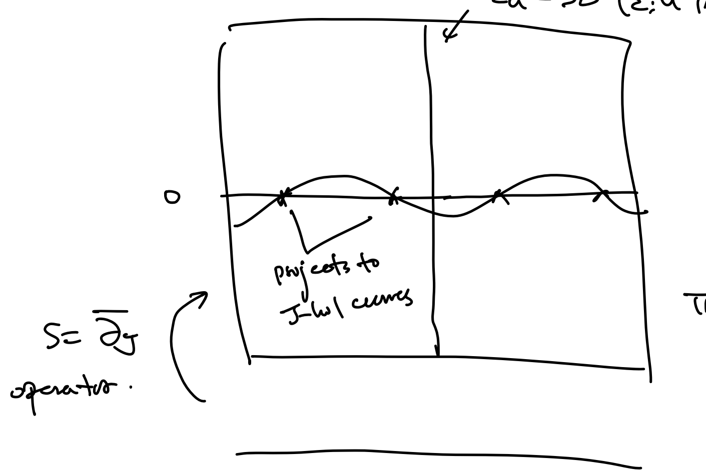
$$\Rightarrow \bar{\partial}_J u \in \Omega^{0,1}(\Sigma; u^*TM)$$

depending on $u$.

Next, let us formulate a vector bundle (over $\mathbb{C}$)




$$\mathcal{E} = \bigcup \{u\} \times \Omega^{0,1}(\Sigma; u^*TM)$$
$$\left(\simeq \Omega^{0,1}(\Sigma; u^*TM)\right)$$

$$\mathcal{B} = C^\infty(\Sigma, M)$$

later we will care about the regularity.

Important observation: $\left\{\begin{array}{l}\text{J-hol curves} \\ u : (\Sigma, j) \to (M, J)\end{array}\right\}$ = section $\bar{\partial}_J \cap$ (0-section of $\mathcal{E} \to \mathcal{B}$)

( moduli space $\mathcal{M}$ = intersection with 0-section)

**= \bar{\partial}_J^{-1}(0)**

<u>Goal</u>: Show that the moduli space $\mathcal{M}$ is a f.d. mfd.

- Recall in differential top: $f : M \to N$ and $S \overset{\text{submfd}}{\subset} N$, $f \pitchfork S$ if $\forall\, p \in M$ with $f(p) \in S$ s.t.

$$f_*(p)(T_pM) + T_{f(p)}S = T_{f(p)}N$$

If so, then $f^{-1}(S)$ is a submfd of $M$ with $\operatorname{codim} f^{-1}(S) = \operatorname{codim} S$.

Naive application to our setting:

$\bar\partial_J : B = C^\infty(\Sigma, M) \longrightarrow \mathcal{E}$, $\quad \mathcal{O}_B \subset \mathcal{E}$ $\quad$ check that

$\forall\, u \in B$ s.t. $\bar\partial_J u \in \mathcal{O}_B$ (i.e. $u$ is J-hol), we have

$$(\bar\partial_J)_*(u)(T_u B) + T_{(u,0)}\mathcal{O}_B = T_{(u,0)}\mathcal{E} \quad (*)$$

$\Longrightarrow$ $\bar\partial_J^{-1}(\mathcal{O}_B)\ (= \mathcal{M})$ is a submfd of $\mathcal{E}$.

$(*)$ rewrites as $\quad (\bar\partial_J)_*(u)(T_u B) + T_u B = T_{(u,0)}\mathcal{E}\ (= T_u B \oplus \mathcal{E}_u)$

$\Longleftrightarrow$ Image of $(\bar\partial_J)_*(u)$ is complementary to $T_u B$ in $T_{(u,0)}\mathcal{E}$.

$\Longleftrightarrow$ $$\underbrace{\Omega^0(\Sigma, u^*TM) = T_u B \xrightarrow{(\bar\partial_J)_*(u)} T_{(u,0)}\mathcal{E} \xrightarrow{\pi_u} \mathcal{E}_u}_{\text{linearization of } (\bar\partial_J)_* \text{ at } u} = \Omega^{0,1}(\Sigma, u^*TM)$$

is surjective.

Prop Linearization is surjective for every $u \in \mathcal{M}$ $\Rightarrow$ $\mathcal{M}$ is a mfd.

- $\mathcal{E} \to B$ is a vector bundle (i.e. $\exists$ local trivialization) //

Let us first work out the local model of this linearization

Recall, locally in coordinate $(s,t)$ where $j\partial_s = \partial_t$

$$\bar\partial_J(u) = \underbrace{\tfrac{1}{2}(\partial_s u + J(u)\partial_t u)ds + \tfrac{1}{2}(\partial_t u - J(u)\partial_s u)dt}_{\in\, \Omega^{0,1}(\mathbb{C}, u^*\mathbb{R}^{2n})}$$

Take $\xi \in \Gamma(u^*\mathbb{R}^{2n})$

the parallel transport is trivial in this case

$$\underbrace{\left(\bar\partial_{J_*}(u)\right)(\xi)}_{\text{derivative in direction } \xi} = \lim_{h\to 0} \frac{\bar\partial_J(u+h\xi) - \bar\partial_J(u)}{h}$$

$$= \frac{1}{2} \lim_{h\to 0} \overbrace{\frac{\left(\partial_s(u+h\xi) + J(u+h\xi)\,\partial_t(u+h\xi)\right) - \left(\partial_s u + J(u)\partial_t u\right)}{h}}^{A}\, ds$$

$$+ \frac{1}{2} \lim_{h\to 0} \overbrace{\frac{\left(\partial_t(u+h\xi) - J(u+h\xi)\,\partial_s(u+h\xi)\right) - \left(\partial_t u - J(u)\partial_s u\right)}{h}}^{B}\, dt$$

For part A:

$$\lim_{h\to 0} \frac{\partial_s(h\xi) + \left(J(u+h\xi)\,\partial_t(u+h\xi) - J(u)\,\partial_t u\right)}{h}$$

$$= \partial_s\xi + \left(\partial_\xi J\right)(u)\cdot\partial_t u + J(u)\,\partial_t\xi.$$

directional derivative of $J$ at $u$ along $\xi$

For part B.

$$\lim_{h\to 0} \frac{\partial_t(h\xi) - \left(J(u+h\xi)\,\partial_s(u+h\xi) - J(u)\,\partial_s u\right)}{h}$$

$$= \partial_t\xi - \left(\partial_\xi J\right)(u)\cdot\partial_s u - J(u)\,\partial_s\xi$$

directional derivative of $J$ at $u$ along $\xi$

$$\Rightarrow \left(\bar\partial_J\right)_*(u)(\xi) = \frac{1}{2}\left(\underline{\partial_s\xi} + (\partial_\xi J)(u)\cdot\partial_t u + \underline{J(u)\partial_t\xi}\right) ds$$

$$+ \frac{1}{2}\left(\underline{\partial_t\xi} - (\partial_\xi J)(u)\cdot\partial_s u - \underline{J(u)\,\partial_s\xi}\right) dt$$

$$= \bar\partial_J \xi + \frac{1}{2}\Big((\partial_\xi J)(u)\,\partial_t u\, ds - (\partial_\xi J)(u)\,\partial_s u\, dt\Big)$$

Since $u$ is $J$-hol, we have $J(u)\partial_t u = -\partial_s u \Rightarrow \partial_t u = J(u)\partial_s u$

Claim: $\underbrace{(J\partial_\xi J)(u)\,\partial_J u}_{\text{this is a closed formula independent of the loc. cor.}} = (\partial_\xi J)(u)\,\partial_s u\, dt - (\partial_\xi J)(u)\,\partial_t u\, ds$

($\partial_J u$: (1,0)-part of $du$)

Pf of claim: $(J\partial_\xi J)(u)\,\partial_J u = (-\partial_\xi J\cdot J)(u)\,\partial_J u$

$$= \frac{1}{2}(-\partial_\xi J)(u)\Big(J(u)(\partial_s u - J(u)\partial_t u)\,ds + J(u)(\partial_t u + J(u)\partial_s u)\,dt\Big)$$

$$= \frac{1}{2}(-\partial_\xi J)\Big((J(u)\partial_s u + \partial_t u)\,ds + (J(u)\partial_t u - \partial_s u)\,dt\Big)$$

$$= (-\partial_\xi J)(\partial_t u\, ds - \partial_s u\, dt)$$

$$= (\partial_\xi J)(\partial_s u\, dt - \partial_t u\, ds) \qquad \square$$

$\Rightarrow$ The local expression of the linearization of $\bar\partial_J$ at $\underset{\uparrow\, J\text{-hol}}{u}$ along $\xi$ is given by

$$\big((\bar\partial_J)_*(u)\big)(\xi) = \bar\partial_J \xi - \frac{1}{2}(J\partial_\xi J)(u)\,\partial_J u$$

Now, let's work more globally (with the help of a connection on $\underset{B}{\overset{\mathcal{E}}{\downarrow}}$).

For $(M,J)$, by prop above, take an affine connection $\tilde\nabla$ on $M$ s.t. $\tilde\nabla J = 0$.

This $\tilde{\nabla}$ helps to construct a parallel transport as follows:

$\mathcal{E}_u = \{u\} \times \Omega^{0,1}(\Sigma, u^*TM)$ $\quad\longrightarrow\quad$ $\mathcal{E}_v = \{v\} \times \Omega^{0,1}(\Sigma, v^*TM)$

$\Omega^{0,1}(\Sigma, u^*TM)$ is the anti-holomorphic part of $\mathrm{Hom}(T\Sigma, u^*TM)$

$$T_z\Sigma \xrightarrow{\varphi(z)} T_{u(z)}M \xrightarrow[\text{assume } u, v \text{ are sufficiently close}]{P_\gamma^{\tilde{\nabla}}} T_{v(z)}M$$

where $\gamma$ connects $v(z)$ and $u(z)$ (say geodesic)

$$P_\gamma^{\tilde{\nabla}} \circ \varphi(z) \in \text{anti-hol part of } \mathrm{Hom}(T_z\Sigma, v^*TM) \quad \text{b/c } \tilde{\nabla}J = 0.$$

let $z$ varies

$\Longrightarrow$

- we obtain a $\mathbb{C}$-linear iso between $\mathcal{E}_u$ and $\mathcal{E}_v$.
(and it implies the desired local trivialization for $\begin{smallmatrix}\mathcal{E}\\ \downarrow \\ \mathcal{B}\end{smallmatrix}$).

- it induces a connection $\overline{\nabla}$ on $\begin{smallmatrix}\mathcal{E}\\ \downarrow \\ \mathcal{B}\end{smallmatrix}$ so that we can differentiate sections in $\Gamma(\mathcal{B}, \mathcal{E})$, in particular $\bar{\partial}_J$. //

## 4. Compute linearization of $\bar{\partial}_J$

Fix $\tilde{\nabla}$ as above, $J$-hol $u \in C^\infty(\Sigma, M)$, and $\xi \in T_u C^\infty(\Sigma, M)$
$= \Gamma(u^*TM)$

By def

$$(\tilde{\nabla}_\xi \partial_J)(u) := \lim_{h\to 0} \frac{P^{\tilde{\nabla}}_{\gamma(h)} \partial_J(\exp_u(h\xi)) - \partial_J(u)}{h}$$

$\gamma$ is a path seen by evaluating it at each pt $z \in \Sigma$

$$= \frac{d}{dh}\Big|_{h=0} P^{\tilde{\nabla}}_{\gamma(h)} \partial_J(\exp_u(h\xi))$$

$$= \frac{1}{2} \frac{d}{dh}\Big|_{h=0} P^{\tilde{\nabla}}_{\gamma(h)} \left(\exp_u(h\xi)_* + J \cdot \exp_u(h\xi)_* \cdot j\right)$$

P^{\tilde{\nabla}}_{\gamma(h)}

$$= \frac{1}{2}\left(\frac{d}{dh}\Big|_{h=0} P^{\tilde{\nabla}}_{\gamma(h)} (\exp_u(h\xi)_*) + \tilde{\nabla}(J \cdot \exp_u(h\xi)_* \cdot j)\right)$$

$\tilde{\nabla} J = 0$

$$= \frac{1}{2}\left(\frac{d}{dh}\Big|_{h=0} P^{\tilde{\nabla}}_{\gamma(h)} (\exp_u(h\xi))_* \right) + J\left(\frac{d}{dh}\Big|_{h=0} P^{\tilde{\nabla}}_{\gamma(h)} (\exp(h\xi)_* \cdot j)\right)$$

it is not easy to calculate it based on $\tilde{\nabla}$.

By standard computation, we know

the Levi-Civita connection on $(M, J, g)$

$$\frac{d}{dh}\Big|_{h=0} P^{\nabla}_{\gamma(h)} (\exp(h\xi)_* \cdot j) = \nabla \xi \quad \text{(b/c torsion free)}$$

no j

Investigate the difference between $P^{\nabla}$ and $P^{\tilde{\nabla}}$.

Recall for any $Y \in \Gamma(TM)$, $\tilde{\nabla}_\xi Y - \nabla_\xi Y = -\frac{1}{2} J (\nabla_\xi J)(Y)$

$$\Rightarrow \frac{d}{dt}\Big|_{h=0} P^{\tilde{\nabla}}_{\gamma(h)} (\exp_u(h\xi)_*) = \nabla \xi - \frac{1}{2} J (\nabla_\xi J)(u_*)$$

$$\Rightarrow \frac{d}{dt}\Big|_{h=0} P^{\tilde{\nabla}}_{\gamma(h)} (\exp_u(h\xi)_* \cdot j) = \nabla_{j(\cdot)} \xi - \frac{1}{2} J (\nabla_\xi J)(u_* \circ j)$$

Therefore,

$$\left(\widetilde{\nabla}_{\xi}\bar{\partial}_J\right)(u) = \frac{1}{2}\left(\nabla\xi - \frac{1}{2}J\left(\nabla_\xi J\right)(u_*)\right) + \frac{1}{2}J(u)\left(\nabla_{j(\cdot)}\xi - \frac{1}{2}J\left(\nabla_\xi J\right)(u_*\cdot j)\right)$$

$J(\nabla_\xi J) = -(\nabla_\xi J)J$

$$= \frac{1}{2}\left(\nabla\xi + J\nabla_{j(\cdot)}\xi\right) + \frac{1}{4}\left((\nabla_\xi J)(u_*\cdot j + J\cdot u_*)\right)$$

$$= \frac{1}{2}\left(\nabla\xi + J\nabla_{j(\cdot)}\xi\right) + \frac{1}{2}(\nabla_\xi J)\left(\partial_J u\right)\cdot j$$

$u_*\cdot j + J\cdot u_* = (u_* - J\cdot u_*\cdot j)\cdot j = (\partial_J u)\cdot j.$

**Prop** $2\left(\widetilde{\nabla}\bar{\partial}_J\right)\overset{J\text{-hol}}{(u)}$ defines an operator $\Gamma(u^*TM) \longrightarrow \Omega^{0,1}(\Sigma, u^*TM)$ by

$$\xi \longmapsto \nabla\xi + J\nabla\xi\cdot j + \nabla_\xi J\cdot u_*\cdot j$$

cf. (2.1) in [Wen]

$u_* = \partial_J u + \bar{\partial}_J u = \partial_J u.$

**Rmk** Usually one uses $D_u$ to represent $2(\widetilde{\nabla}\bar{\partial}_J)(u)$. and it is called the linearized Cauchy-Riemann operator at $u$.

Here is a more general def:

**Def** (Def 2.1 in [Wen]) Let $\begin{matrix}(E,J)\\ \downarrow\\ (\Sigma,j)\end{matrix}$ be a cpx v.b. A (real) linear C-R type operator on $E$ is real-linear 1st order differential operator

$$D : \Gamma(E) \longmapsto \Omega^{0,1}(\Sigma; E)$$

s.t. $D(fs) = (\bar{\partial}f)\otimes s + fDs$ for any $f\in C^\infty(\Sigma,\mathbb{R})$ and $s\in\Gamma(E)$.

Here $\bar{\partial}f = df + J\,df\circ j$.

One can verify that $D_u$ is a linear C-R type operator on $\begin{array}{c}(u^*TM, J)\\ \downarrow \\ (\Sigma, j)\end{array}$

$$D_u(f\xi) = \nabla(f\xi) + J\,\nabla(f\xi)\cdot j + \nabla_{f\xi} J \cdot u_* \cdot j$$

$$= df \otimes \xi + f\nabla\xi + J\left(df \otimes \xi + f\nabla\xi\right)\cdot j + f\,\nabla_\xi J \cdot u_* \cdot j$$

$$= \left(df + J\cdot df \cdot j\right)\otimes \xi + f\left(\nabla\xi + J\cdot\nabla\xi\cdot j + \nabla_\xi J \cdot u_* \cdot j\right)$$

$$= \bar\partial f \otimes \xi + f(D_u\xi) \quad \checkmark$$

Finally we have the following fact.

<u>FACT</u>: $\mathbb{D}$ is $\mathbb{C}$-linear iff $\begin{array}{c}E\\ \downarrow\\ M\end{array}$ is a <u>holomorphic</u> vector bundle.

Leibniz rule holds for $f \in C^\infty(\Sigma;\mathbb{C})$.

(i.e. transition maps are holomorphic)

//