Then

$$\Omega^{p,q}(M) := \{\text{sections of bundle } (*) \text{ above}\}$$

is a module over $C^\infty(M;\mathbb{C})$; each $\omega \in \Omega^{p,q}(M)$ called a (p,q)-form.

Ex (Σ_g, j) Riem surface. locally choose coordinate (s,t), then

$$T_*M = \mathrm{span}_{\mathbb{R}}\langle \partial_s, \partial_t \rangle \Rightarrow T^*M = \mathrm{span}_{\mathbb{R}}\langle ds, dt \rangle$$

$$\nwarrow j\partial_s = \partial_t \qquad\qquad \nwarrow jds = -dt$$

$$\Rightarrow (T^*M)_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\langle ds, dt \rangle$$

Then $\Omega^{1,0}(M) = \mathrm{span}_{\mathbb{C}}\langle ds - \sqrt{-1}\, j\, ds \rangle = \mathrm{span}_{\mathbb{C}}\langle ds + \sqrt{-1}\, dt \rangle$

$\Omega^{0,1}(M) = \mathrm{span}_{\mathbb{C}}\langle ds + \sqrt{-1}\, j\, ds \rangle = \mathrm{span}_{\mathbb{C}}\langle ds - \sqrt{-1}\, dt \rangle$

$\Rightarrow \Omega^0(M) = \Omega^{0,0}(M) = \{\mathbb{C}\text{-valued smooth fcns on } M\}$

$\Omega^1(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M) = \cdots$ (above)

$\Omega^2(M) = \overset{0}{\cancel{\Omega^{2,0}(M)}} \oplus \Omega^{1,1}(M) \oplus \overset{0}{\cancel{\Omega^{0,2}(M)}}$

$= \mathrm{span}_{\mathbb{C}}\langle (ds + \sqrt{-1}\, dt) \wedge (ds - \sqrt{-1}\, dt) \rangle$

$= \mathrm{span}_{\mathbb{C}}\langle ds \wedge dt \rangle \leftarrow$ of cource!

//

2. Changing coefficient (of sections)

Given a vector bundle $\begin{matrix} F \\ \downarrow \\ M \end{matrix}$ over field $\Bbbk$, any other v.b. $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ can helps to change the coefficents of $s : M \to F$.

w.r.t any basis, $\{s_1, \cdots, s_n\}$ $s = f_1 s_1 + \cdots + f_n s_n$ for $f_i \in C^\infty(M; \mathbb{K})$

Namely, consider tensor bundle $\begin{matrix} F\otimes E \\ \downarrow \\ M \end{matrix}$, a section is a combination of $\theta \otimes s$ where $\theta \in \Gamma(M; F)$ and $s \in \Gamma(M, E)$.

$\Rightarrow$ Take $F = T^*M$ or $\Lambda^k M$, then

$$(\theta \otimes s)(\underset{\substack{\Uparrow \\ TM \text{ or } TM^{\otimes k}}}{v}) = f \cdot s \in \Gamma(M; E).$$

$\Rightarrow$ Any $\omega \in \Gamma(T^*M \otimes E)$ can be written as $\omega = \underbrace{\theta_1}\underbrace{\otimes s_1} + \cdots\ \theta_n \otimes \underbrace{s_m}$

E-valued section.
(or E-coefficient section)

This also holds for $\Gamma(\Lambda^k M \otimes E)$.

Notation. $\Gamma(\Lambda^k M \otimes E) =: \Omega^k(M; E)$.

<u>Rmk</u> $\Omega^k(M) = \Omega^k(M; M \times \mathbb{R})$ ← trivial rank-1 bundle over M.

<u>Rmk</u> $\theta \in \Omega^k(M)$, $\omega \in \Omega^l(M; E) \mapsto \theta \wedge \omega \in \Omega^{k+l}(M; E)$

so $\Omega^\bullet(M; E)$ is a module over $\Omega^\bullet(M)$.

In general, $\Omega^\bullet(M; E)$ does <u>NOT</u> admit any algebra str.

(e.g. E = a Lie algebra, then $\Omega^\bullet(M; E)$ is an algebra).

In this spirit, one can write many things as bundle-valued section.

Ex · $J \curvearrowright TM \Rightarrow J \in \Gamma(\mathrm{Hom}_{\mathbb{R}}(TM, TM)) = \Gamma(\mathrm{End}(TM))$

· $\begin{matrix} E \\ \downarrow \\ M \end{matrix} \rightsquigarrow \begin{matrix} E^* \\ \downarrow \\ M \end{matrix}$ dual bundle $\rightsquigarrow \begin{matrix} E^*\otimes E^* \\ \downarrow \\ M \end{matrix}$

$\Rightarrow$ any metric g on $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ is $\in \Gamma(E^*\otimes E^*)$

· A connection ∇ on $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ is a map

$$\nabla : \Gamma(E) \longrightarrow \Omega^1(M;E)$$

satisfying $\underbrace{\nabla(fs) = \overset{\Omega^1(M)}{\underset{\Omega^1(M)}{df}}\otimes s + f\nabla s.}$

Leibniz rule

digest of this definition

① In particular, if $E = TM$, then $\nabla X \in \Omega^1(M;TM)$

$\Rightarrow$ inputting $Y \in \Gamma(TM)$, we get $\underbrace{(\nabla X)(Y)}_{\text{more standard: } \nabla_Y X} \in \Gamma(TM)$ $\leftarrow$ learned in differential mfd class.

② One can extend to $\nabla : \Omega^k(M;E) \longrightarrow \Omega^{k+1}(M;E)$. by induction.

Define $\nabla : \Omega^1(M;E) \longrightarrow \Omega^2(M;E)$

$$\nabla(\underset{\Omega^1(M)}{\theta}\otimes s) := d\theta\otimes s \pm \underbrace{\theta\wedge \underbrace{\nabla s}_{\in\Omega^1(M;E)}}_{\in\Omega^2(M;E)}$$

Rmk One can check that ∇^2 is always a tensor (i.e. $\nabla^2(f-) = f\nabla^2(-)$)
This is related with the curvature of a connection on $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$.

③ $\begin{matrix} E = M\times\mathbb{R}^k \\ \downarrow \\ M. \end{matrix} \circlearrowleft s \simeq (f_1, \cdots, f_k)$ one can take $\nabla(s) = (df_1, \cdots, df_k)$

//

$$\begin{tikzcd} \mathrm{End}(TM) \arrow[d] \\ M \end{tikzcd} \;, \; \nabla \quad \Rightarrow \quad \nabla J \in \Omega^1(M; \mathrm{End}(TM)) \rightsquigarrow (\nabla_X J)(-)$$

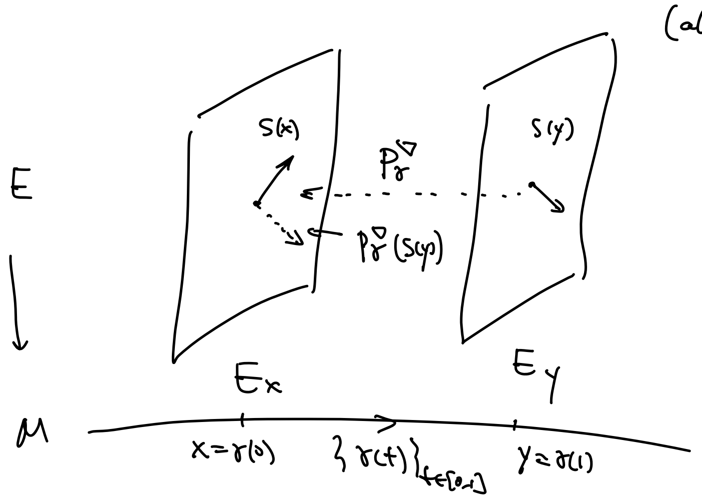
$$\begin{tikzcd} E^* \otimes E^* \arrow[d] \\ M \end{tikzcd} \;, \; \nabla \quad \Rightarrow \quad \nabla g \in \Omega^1(M; E^* \otimes E^*) \rightsquigarrow \nabla_X g(-,-)$$

Recall both $\nabla_X J$, $\nabla_X g$ have the a unified meaning: directional der along the direction X.

$\Rightarrow$ ∇ helps us to "differentiate" a section.

one needs parallel transport.

(allows different fiber to communicate)



$\gamma(0) = x$ and $\gamma(1) = y$.

A parallel transport (of $\underset{\overset{\uparrow}{\text{input data}}}{\cdots\cdots}$) is a linear iso $P^\nabla_\gamma : E_y \to E_x$.

recall its def ✓ Given $p \in E_y$, solve a section $s \in E$ satisfying $\nabla_{\dot\gamma(t)} s = 0$ with the initial cond. $s(\gamma(0)) = s(y) = p$ (uniqueness by ODE) and then

$$P^\nabla_\gamma(p) := s(\gamma(1)).$$

Then the derivative of a section $s \in \Omega^0(M; E)$ at pt $x \in M$ in direction $\xi \in T_x M$ is defined by

$$(\nabla_\xi s)(x) := \lim_{h \to 0} \frac{P_\gamma^\nabla (s(\exp_x(h\xi)) - s(x)}{h}$$

One can show this is ind of the choice of connecting path γ (connecting $\exp_x(h\xi)$ and x).

Back to ∇J and ∇g: How to obtain these connections?

<u>Ex</u> Any (affine) connection ∇ on $\begin{matrix} TM \\ \downarrow \\ M \end{matrix}$ induces a connection ∇ on $\begin{matrix} End(TM) \\ \downarrow \\ M \end{matrix}$ by

$$(\nabla J)(Y) := \nabla(JY) - J(\nabla Y) \qquad (*)$$

($\Leftrightarrow \nabla(JY) = (\nabla J)(Y) + J(\nabla Y)$ notationally beautiful but could be very misleading)

For $X, Y \in \Gamma(M, TM)$, $(\underbrace{\nabla_X J}_{\in \Omega^1(M, End(TM))})(Y) \in TM$ and $(*)$ above means

$$(\nabla_X J)(Y) = \nabla_X(JY) - J(\nabla_X Y)$$

RHS is well-defined since ∇ is an affine connection.

One verifies that ∇J defined above is a connection

$$
\begin{aligned}
(\nabla_X(fJ))(Y) &= \nabla_X(fJ(Y)) - fJ(\nabla_X Y) \\
&= f(\nabla_X(JY)) + df(X)\cdot J(Y) - fJ(\nabla_X Y) \\
&= f(\nabla_X(JY) - J\nabla_X Y) + df(X).J(Y) \\
&= f\cdot(\nabla_X J)(Y) + df(X)\cdot J(Y)
\end{aligned}
$$

Omitting X, Y, we get $\nabla(fJ) = f\cdot\nabla J + df\otimes J$ ✓.

<u>Ex</u> Any affine connection induces a connection ∇ on $\begin{matrix} E^*\otimes E^* \\ \downarrow \\ M. \end{matrix}$

by $\nabla g(s_1,s_2) := g(\nabla s_1, s_2) + g(s_1, \nabla s_2)$.

This is incorrect (pointed by Yichen YAO in class): it should be \nabla g(s_1, s_2) := \nabla (g(s_1, s_2)) - g(\nabla s_1, s_2) - g(s_1, \nabla s_2).
Here, the first term on the RHS is simply the directional derivative of function g(s_1, s_2).

Recall, given a Riem mfd (M,g), the Levi-Civita connection is an affine connection s.t. $\nabla g = 0$ (and $E^* = T^*M$) and torsion-free.

//.

<u>Prop</u> Given any a.c. mfd (M,J), $\exists$ an affine connection $\tilde\nabla$ s.t. $\tilde\nabla J \equiv 0$.

<u>Pf</u>. $(M,J) \rightsquigarrow$ a J-compatible Riem metric g $\quad g(J(\cdot),J(\cdot)) = g(\cdot,\cdot)$
(working locally + p.o.u)

$\rightsquigarrow \nabla$ the Levi-Civita connection of (M,g)

$\rightsquigarrow \tilde\nabla Y := \nabla Y - \frac{1}{2}J(\underset{\nwarrow}{\nabla J})(Y)$ induced connection on $\begin{matrix}\mathrm{End}(TM)\\ \downarrow \\ M.\end{matrix}$

Verify that $\tilde{\nabla} J \equiv 0$. For any $Y \in \Gamma(M, TM)$,

$$(\tilde{\nabla} J)(Y) = \tilde{\nabla}(JY) - J\tilde{\nabla} Y$$

$$= \left(\nabla(JY) - \tfrac{1}{2} J(\nabla J)(JY)\right) - J\left(\nabla Y - \tfrac{1}{2} J(\nabla J)(Y)\right)$$

$$= \nabla(JY) - \tfrac{1}{2} J(\nabla J)(JY) - J(\nabla Y) - \tfrac{1}{2}(\nabla J)(Y)$$

$$= (\nabla J)(Y) - \tfrac{1}{2} J(\nabla J)(JY) - \tfrac{1}{2}(\nabla J)(Y)$$

Observe that $J(\nabla J) = -(\nabla J)J$ (b/c $J^2 = -\mathbb{1}$). Then

$$\cdots = (\nabla J)(Y) + \tfrac{1}{2}(\nabla J)J(JY) - \tfrac{1}{2}(\nabla J)(Y) = 0. \quad \square$$

The prove above also implies that $\tilde{\nabla} g \equiv 0$.

$$(\tilde{\nabla} g)(X, Y) = g(\tilde{\nabla} X, Y) + g(X, \tilde{\nabla} Y)$$

$$= g\left(\nabla X - \tfrac{1}{2} J(\nabla J)(X), Y\right) + g\left(X, \nabla Y - \tfrac{1}{2} J(\nabla J)(Y)\right)$$

$$= \underbrace{g(\nabla X, Y) + g(X, \nabla Y)}_{=0} - \tfrac{1}{2} g(J(\nabla J)(X), Y) - \tfrac{1}{2} g(X, J(\nabla J)(Y))$$

Due to the incorrect definition of the induced connection acting on g above, the verification of the claim that \tilde{\nabla} g \equiv 0 here is not correct. However, the conclusion is still true, and we leave this verification as a homework problem.

Observe that $g(J(\nabla J)(X), Y) = -g((\nabla J)(X), J(Y))$

$$= -g(\nabla(JX) - J(\nabla X), J(Y))$$

$$= g(\nabla X, Y) - g(\nabla(JX), JY)$$

$$\cdots \overset{X(-2)}{=} g(\nabla X, Y) - g(\nabla(JX), JY) + g(\nabla Y, X) - g(\nabla(JY), JX)$$

$$= \underbrace{g(\nabla X, Y) + g(X, \nabla Y)}_{=0} - \big(\underbrace{g(\nabla(JX), JY) + g(JX, \nabla(JY))}_{=0}\big).$$

<u>Rmk</u> $\tilde\nabla$ is <u>not</u> nec torsion free! $\tilde\nabla$ is torsion free $\iff$ J is integrable.

<u>Rmk</u> Given a symp mfd (M,ω) $\rightsquigarrow$ ω-compatible J

$\rightsquigarrow$ $g := \omega(-, J-)$ a Riem metric

$\rightsquigarrow$ ∇ Levi-Civita connection of g

$\rightsquigarrow$ $\tilde\nabla$ s.t. $\tilde\nabla J = \tilde\nabla g \equiv 0$.

$\rightsquigarrow$ $\tilde\nabla$ is the Levi-Civita connection of g

$\iff$ M admits a Kähler str.

characterization of the Kähler str from connections

+

Kähler $\subset$ symplectic.

$\uparrow$

the difference is a "torsion" condition

3. Cauchy-Riemann equation

$u : (\Sigma, j) \longrightarrow (M, J)$ recall we have defined

$$\bar\partial_J u = \frac{1}{2}(u_* + J \cdot u_* \circ j)$$

For each pt $z \in \Sigma$, $(\bar\partial_J u)_z \in \mathrm{Hom}_{\mathbb{R}}(T_z\Sigma, T_{u(z)}M)$

$\Rightarrow$ $\bar\partial_J u \in \Gamma(\mathrm{Hom}(T\Sigma, u^*TM))$ where u^*TM is the pullback bundle of $\begin{smallmatrix} TM \\ \downarrow \\ M \end{smallmatrix}$ under u.

$$\begin{tikzcd} u^*TM \arrow[r] \arrow[d] & TM \arrow[d] \\ \Sigma \arrow[r, "u"] & M \end{tikzcd}$$

$$(u^*TM)_z = \{(z, v) \mid v \in T_{u(z)}M\}$$
$$= \{\text{tangent vector along the image of } u\}.$$

Note that $\mathrm{Hom}_{\mathbb{R}}(T\Sigma, u^*TM) \simeq T^*\Sigma \otimes_{\mathbb{R}} u^*TM$

$(\Sigma, j) \Rightarrow j$ is an a.c.s on $T\Sigma$

$\Rightarrow$ a.c.s (still denoted by) j on $T^*\Sigma$

$\Rightarrow (T^*\Sigma)_{\mathbb{C}}$ complexification

$\Rightarrow (T^*\Sigma)_{\mathbb{C}} \otimes_{\mathbb{C}} u^*TM$ is a $\underline{\text{complex bundle}}$ (over Σ)

each fiber admits a cpx str.

$$(a + \sqrt{-1}\, b)(\underset{\underset{(u^*TM)_z}{\cap}}{v}) := a \cdot v + J(bv)$$

Then $\bar{\partial}_J u \in \Gamma((T^*\Sigma)_{\mathbb{C}} \otimes_{\mathbb{C}} u^*TM) = \Omega^1(\Sigma; u^*TM)$

over $\mathbb{C}$

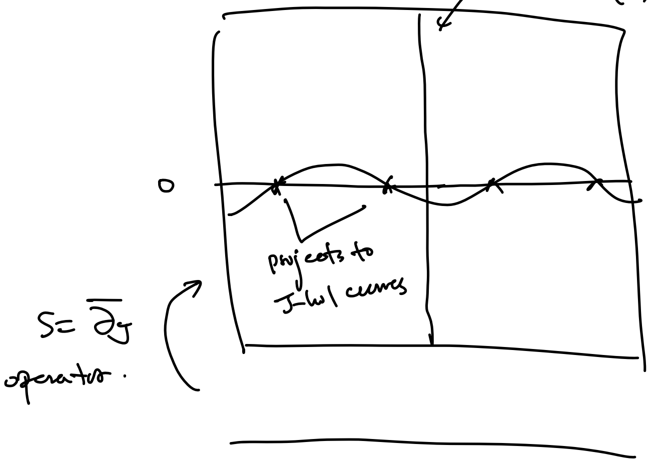
Moreover,

$$\begin{aligned} j^*(\bar{\partial}_J u)(v) &= (\bar{\partial}_J u)(j(v)) \\ &= \tfrac{1}{2}(u_* + J \cdot u_* \cdot j)(j(v)) \\ &= \tfrac{1}{2}(u_* \cdot j - J \cdot u_*)(v) \\ &= -\tfrac{1}{2} J (u_* - J \cdot u_* \cdot j)(v) \\ &= -J(\bar{\partial}_J u)(v) \end{aligned}$$

$\Rightarrow \bar{\partial}_J u \in \Omega^{0,1}(\Sigma; u^*TM)$ ⟵ depending on u.

Next, let us formulate a vector bundle (over $\mathbb{C}$)

$\mathcal{E}_u = \Omega^{0,1}(\Sigma; u^*TM)$



$\mathcal{E} = \bigcup \{u\} \times \Omega^{0,1}(\Sigma; u^*TM)$

$(\simeq \Omega^{0,1}(\Sigma; u^*TM))$

$\downarrow \pi$

$\mathcal{B} = C^\infty(\Sigma, M)$

later we will care about the regularity.

Important observation: $\left\{ \begin{matrix} J\text{-hol curves} \\ u: (\Sigma, j) \to (M, J) \end{matrix} \right\}$ = section $\bar{\partial}_J \cap$ (0-section of $\mathcal{E} \to \mathcal{B}$)

(moduli space $\mathcal{M}$ = intersection with 0-section)

<u>Goal</u>: Show that the moduli space $\mathcal{M}$ is a f.d. mfd.

- Recall in differential top: $f: M \to N$ and $S \overset{\text{submfd}}{\subset} N$, $f \pitchfork S$ if $\forall$ $p \in M$ with $f(p) \in S$ s.t.

$$f_*(p)(T_pM) + T_{f(p)}S = T_{f(p)}N$$

If so, then $f^{-1}(S)$ is a submfd of M with codim $f^{-1}(S)$ = codim S.