Cauchy-Rieman equation and its linearization

1. Complexification

V vector space over $\mathbb{R}$

complexification of V is $V \otimes_{\mathbb{R}} \mathbb{C} = \{ v_1 + \sqrt{-1}\, v_2 \mid v_i \in V \}$

$$\underset{\text{category}}{\left\{ \begin{array}{l} \text{vector spaces} \\ \text{over } \mathbb{R} \text{ in } \dim_{\mathbb{R}} = n \end{array} \right\}} \xrightarrow{\text{complexification}} \underset{\text{category}}{\left\{ \begin{array}{l} \text{vector spaces} \\ \text{over } \mathbb{C} \text{ in } \dim_{\mathbb{C}} = n \end{array} \right\}}$$

is a functor $f : V \to W \longrightarrow f : V \otimes_{\mathbb{R}} \mathbb{C} \to W \otimes_{\mathbb{R}} \mathbb{C}$

$$v_1 + \sqrt{-1}\, v_2 \to f(v_1) + \sqrt{-1}\, f(v_2)$$

Notation: $V \otimes_{\mathbb{R}} \mathbb{C}$ denoted by $V_{\mathbb{C}}$.

- Compare with V, $V_{\mathbb{C}}$ admits an extra operation: complex conjugation

$$\overline{} : V_{\mathbb{C}} \circlearrowleft \qquad v_1 + \sqrt{-1}\, v_2 \xrightarrow{\overline{}} v_1 - \sqrt{-1}\, v_2 .$$

- An inner product $(\,,\,)$ on V induces a Hermitian inner product on $V_{\mathbb{C}}$, by

$$\langle v_1 + \sqrt{-1}\, v_2 ,\, w_1 + \sqrt{-1}\, w_2 \rangle \underset{\uparrow}{=} (v_1, w_1) + (v_2, w_2) + \sqrt{-1}\, \big((v_2, w_1) - (v_1, w_2) \big)$$

take complex conjugation of the second input

Now, $J : V \to V$ is called a cpx str if $J^2 = -\mathbb{1}_V$.

By discussion above, complexification induces $J: V_{\mathbb{C}} \to V_{\mathbb{C}}$ by $J(v_1 + \sqrt{-1}\, v_2) = J(v_1) + \sqrt{-1}\, J(v_2)$, linear over $\mathbb{C}$.

$$\Rightarrow \quad V_{\mathbb{C}} = \left(\sqrt{-1}\text{-eigenspace of } J\right) \oplus \left(-\sqrt{-1}\text{-eigenspace of } J\right)$$

$$\overset{\text{notation}}{=} V_{\mathbb{C},(1,0)} \oplus V_{\mathbb{C},(0,1)}$$

$$= \{ v - \sqrt{-1}\, Jv \mid v \in V \} \oplus \{ v + \sqrt{-1}\, Jv \mid v \in V \}$$

Note that elements from $V_{\mathbb{C},(1,0)}$ and $V_{\mathbb{C},(0,1)}$ switch to each other by taking complex conjugation.

(Verify: $J(v - \sqrt{-1}\, Jv) = Jv + \sqrt{-1}\, v = \sqrt{-1}\,(v - \sqrt{-1}\, Jv)$)

//

$$V \xrightarrow{\text{dual}} V^* = \{ \mathbb{R}\text{-linear } \varphi : V \to \mathbb{R} \}.$$

$$\begin{tikzcd} V \arrow[r, "\text{dual}", "\text{over } \mathbb{R}"'] \arrow[d, "\text{cpxify}"'] & V^* \arrow[d, "\text{cpxify}"] \\ V_{\mathbb{C}} \arrow[r, "\text{dual}", "\text{over } \mathbb{C}"'] & V_{\mathbb{C}}^* \end{tikzcd}$$

ambiguity of this notation

Commutativity of this diagram : $(V^*)_{\mathbb{C}} \overset{?}{=} (V_{\mathbb{C}})^*$

By definition:

$$(V^*)_{\mathbb{C}} = \{ \varphi_1 + \sqrt{-1}\, \varphi_2 \mid \varphi_i \in V^* \}$$

$$(V_{\mathbb{C}})^* = \{ \mathbb{C}\text{-linear } \varphi : V_{\mathbb{C}} \to \mathbb{C} \}$$

Prop Consider $r: (V_{\mathbb{C}})^* \to (V^*)_{\mathbb{C}}$ by $r(\varphi) = \varphi|_V$, then r is an isomorphism. In particular $(V_{\mathbb{C}})^* \simeq (V^*)_{\mathbb{C}} (= V^*_{\mathbb{C}})$.

pf If $r(\varphi) = 0 = \varphi|_V$, then $\varphi|_{\sqrt{-1}V} = 0$ b/c φ is $\mathbb{C}$-linear.

$\Rightarrow \varphi = 0$. (b/c $V_{\mathbb{C}} = V + \sqrt{-1}V$) $\square$

Rmk $r^{-1}: (V^*)_{\mathbb{C}} \to (V_{\mathbb{C}})^*$ is defined by

$$\left(r^{-1}(\varphi_1 + \sqrt{-1}\varphi_2)\right)(v_1 + \sqrt{-1}v_2) = \left(\varphi_1(v_1) + \varphi_2(v_2)\right) + \sqrt{-1}\left(\varphi_1(v_2) + \varphi_2(v_1)\right).$$

Conjugation on $(V^*)_{\mathbb{C}}$: $\overline{\varphi_1 + \sqrt{-1}\varphi_2} =: \varphi_1 - \sqrt{-1}\varphi_2$

Conjugation on $(V_{\mathbb{C}})^*$: $\overline{\varphi} := r^{-1}\left(\overline{r(\varphi)}\right)$

$v = v_1 + \sqrt{-1}v_2 \in V_{\mathbb{C}}$

$$\begin{aligned}
\overline{\varphi}(v) &= \overline{\varphi}(v_1 + \sqrt{-1}v_2) \\
&= \left(r^{-1}(\overline{r(\varphi)})\right)(v_1 + \sqrt{-1}v_2) \\
&= \overline{r(\varphi)}(v_1) + \sqrt{-1}\,\overline{r(\varphi)}(v_2) \\
&= \overline{\varphi(v_1)} + \sqrt{-1}\,\overline{\varphi(v_2)} \\
&= \overline{\varphi(v_1) - \sqrt{-1}\varphi(v_2)} \\
&= \overline{\varphi(v_1 - \sqrt{-1}v_2)} = \overline{\varphi(\overline{v})}.
\end{aligned}$$

V^* is important in general b/c one can form tensor algebra or wedge algebra

$V^* \rightsquigarrow \otimes^k V^*$ or $\wedge^k V^*$.

Similarly, one can define $\wedge^k V^*_{\mathbb{C}} = \{$alternating cpx k-linear maps$\}$ //

$$(V^{2n}, J) \rightsquigarrow J \circlearrowright V_{\mathbb{C}}^* \qquad (J(\varphi))(v) := \varphi(J(v))$$

and $J^2 = -\mathbb{1}_{V_{\mathbb{C}}^*}$

$$\Rightarrow \quad V_{\mathbb{C}}^* = V_{\mathbb{C}}^{*,(1,0)} \oplus V_{\mathbb{C}}^{*,(0,1)}$$

$$= \operatorname{span}_{\mathbb{C}}\langle \underbrace{\theta_1, \cdots, \theta_n}_{\text{basis of } (1,0)\text{-type}} \rangle \oplus \operatorname{span}_{\mathbb{C}}\langle \underbrace{\bar\theta_1, \cdots, \bar\theta_n}_{\text{basis of } (0,1)\text{-type}} \rangle$$

$0 \leq p, q \leq n$ $\mathbb{C}$-linear map of (p,q)-type

These basis elements should be those with \bar.

$$\Rightarrow \quad \Lambda^{p,q} V_{\mathbb{C}}^* = \operatorname{span}_{\mathbb{C}}\left\langle \theta_{i_1} \wedge \cdots \wedge \theta_{i_p} \wedge \theta_{j_1} \wedge \cdots \wedge \theta_{j_q} \;\middle|\; \begin{array}{l} 1 \leq i_1 < \cdots < i_p \leq n \\ 1 \leq j_1 < \cdots < j_q \leq n \end{array} \right\rangle$$

$$\Rightarrow \quad \Lambda^k V_{\mathbb{C}}^* = \bigoplus_{\substack{p+q=k \\ 0 \leq p,q \leq n}} \Lambda^{p,q} V_{\mathbb{C}}^*$$

Also, $\theta \in \Lambda^{p,q} V_{\mathbb{C}}^*$, $\omega \in \Lambda^{r,s} V_{\mathbb{C}}^* \Rightarrow \theta \wedge \omega \in \Lambda^{p+r, q+s} V_{\mathbb{C}}^*$.

Prop: V f.d. real vector space of dim $2n$, equipped with an cpx str. then we can defined a bi-graded algebra $\Lambda^{\cdot \cdot} V_{\mathbb{C}}^*$ where each (p,q)-piece has cpx-dim $\binom{n}{p}\binom{n}{q}$.

Move to the mfd setting: (M^{2n}, J) $J: TM \circlearrowleft$ a cpx str.

Then (T^*M, J) helps to form the cpx bundle

$$\Lambda^{p,q}(T^*M)_{\mathbb{C}} \underset{s = \text{section}}{\overset{\pi}{\rightleftarrows}} M \qquad (*)$$

each fiber admits a cpx str.