3. Symplectic embedding

Given two Liouville domain (U, ω_U) and (V, ω_V), a symp emb is an emb $\varphi: U \hookrightarrow V$ s.t. $\varphi^* \omega_V = \omega_U$.

Ex. $(\mathbb{R}^{2n}, \omega_{std}) \left(\simeq (\mathbb{C}^n, \omega_{std})\right)$

$$E(a_1, \cdots, a_n) = \left\{ (z_1, \cdots, z_n) \in \mathbb{C}^n \;\middle|\; \frac{\pi |z_1|}{a_1} + \cdots + \frac{\pi |z_n|^2}{a_n} \le 1 \right\}$$ ← symplectic ellipsoid

$$B^{2n}(r) = E(r, \cdots, r), \qquad Z^{2n}(R) = E(R, \infty, \cdots, \infty)$$

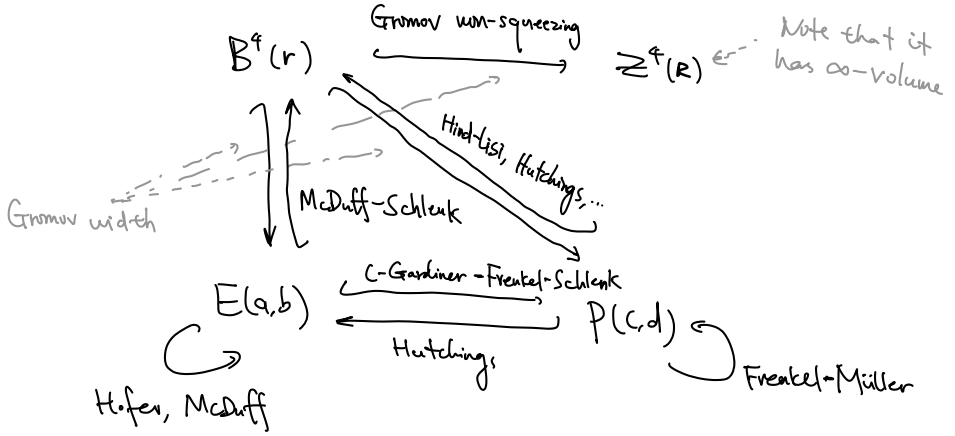
$B^{2n}(r)$: symplectic ball; $Z^{2n}(R)$: symplectic cylinder; $\infty, \cdots, \infty$: no constraint on $z_2, \cdots, z_n$.

$$P(a_1, \cdots, a_n) = \left\{ (z_1, \cdots, z_n) \in \mathbb{C}^n \;\middle|\; \pi |z_1|^2 \le a_1, \cdots, \pi |z_n|^2 \le a_n \right\}$$ ← polydisk

Rmk. $\partial P(a_1, \cdots, a_n)$ is not smooth.

Rmk All cases above admit $\mathbb{T}^n$-action by $(\theta_1, \cdots, \theta_n) \cdot (z_1, \cdots, z_n)$ defined by $(e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n)$. $\Rightarrow$ toric domain

Finding obstructions of embedding is a central topic in symp. geo.



Thm (Gromov) $B^{2n}(r) \hookrightarrow Z^{2n}(R)$ iff $r \leq R$.

Thm $P(a,b) \hookrightarrow P(c,d)$ with $a \leq c$ but $b \geq d$, then $c \geq 2a$.

For $U, V (\subset \mathbb{R}^{2n})$ Liouville domains (w.r.t Y radial v.f.).

← also called star-shaped.

Assume $\varphi(U)$ is also star-shaped, then $V \setminus \text{int}(\varphi(U)) =: W$ satisfies the following properties

- $\partial W = -\partial\varphi(U) \sqcup \partial V$ (with negative orientation)
- Both $\left(\partial\varphi(U), \ker\lambda|_{\partial\varphi(U)}\right)$ and $\left(\partial V, \ker\lambda|_{\partial V}\right)$ are contact wrt ω_{std} when $\lambda = \iota_Y \omega_{std}$.
- $Y \pitchfork \partial V$ pointing outward, $Y \pitchfork \partial\varphi(U)$ pointing inward.

Such $\left(W, \omega_{std}|_W\right)$ is a Liouville cobordism (or exact) from $\partial\varphi(U)$ to ∂V.

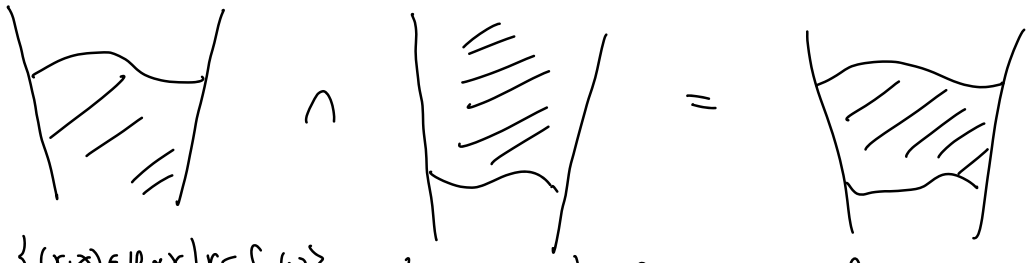
Here, we always assume that a Liouville cobordism is compact.

Rmk More general, if Y is not globally defined, i.e. when $\partial W = -\partial W_- \sqcup \partial W_+$

$\exists\, Y_-$ near ∂W_- and Y_+ near ∂W_+.

Ex $(X, \xi = \ker\alpha) \rightsquigarrow (M = \mathbb{R}\times X,\ \omega = d(e^r\alpha))$

Given any two functions $f_-, f_+ : X \to \mathbb{R}$ s.t. $f_- < f_+$ pointwise



$\{(r,x)\in\mathbb{R}\times X \mid r \le f_+(x)\}$ $\cap$ $\{(r,x)\in\mathbb{R}\times X \mid r \ge f_-(x)\}$ $=$ $W = \{f_-(x) \le r \le f_+(x)\}$

Then W is a Liouville cobordism from $\{r = f_-(x)\}$ to $\{r = f_+(x)\}$.

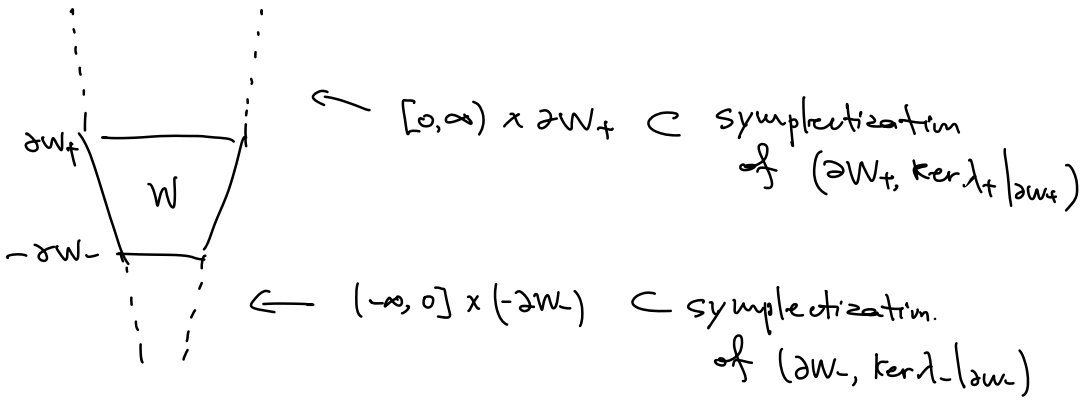
- In particular, a Liouville domain U is a Liouville cobordism W with $\partial W_- = \emptyset$.
- For a contact mfld (X,ξ), a Liouville filling (or an exact filling) is a Liouville domain $(U, \underbrace{\omega}_{d\lambda})$ s.t. $(\partial U, \ker\lambda|_{\partial U}) \cong (X,\xi)$.

Rmk $\nexists$ Liouville cobordism $(W, \omega = d\lambda)$ s.t. $\partial W_+ = \emptyset$.

$$\underbrace{\int_W \omega\wedge\cdots\wedge\omega}_{>0} = \int_W d(\lambda\wedge d\lambda\wedge\cdots\wedge d\lambda)$$
$$= \int_{\partial W_+} \lambda\wedge d\lambda\wedge\cdots\wedge d\lambda - \int_{\partial W_-} \lambda\wedge d\lambda\wedge\cdots\wedge d\lambda$$
$$= 0 - (\text{positive}) < 0 \qquad \to\leftarrow$$

$\Rightarrow$ symplectic cobordism is NOT an equivalence relation. (not symmetric). //

Given a symplectic cobordism (W, ω), one can "complete" it by gluing positive cylindrical end and negative cylindrical end



Ex For a Liouville domain $(U, \omega = d\lambda)$, its completion is a non-cpt exact symp mfld $(\hat{U}, \omega = d\lambda)$

$$E(a_1, \cdots, a_n) \xrightarrow{\text{completion}} (\mathbb{C}^n, \omega_{std})$$

$$(Q, g) \rightsquigarrow U_g^* Q \simeq \{(q,p) \mid \|p\|_{g_q} \leq 1\} \quad \text{unit codisk bundle}$$

$$\xrightarrow{\text{completion}} (T^*Q, \omega_{can})$$

Rmk Completion allows us to work with a non-cpt (exact) symp mfld instead of a mfld with b/d (which is always harder). //