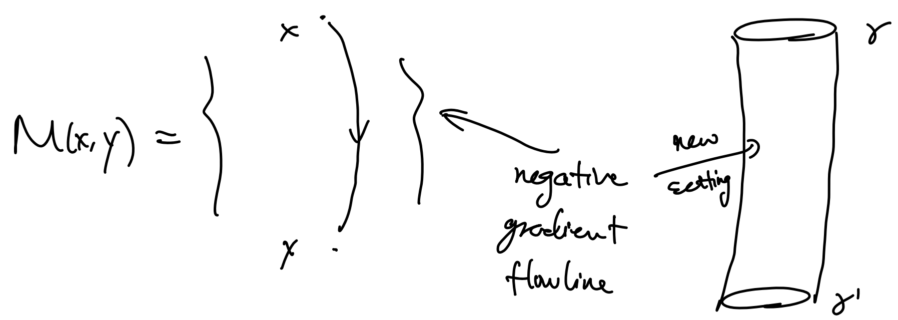


Pick $\{J_t\}_{t\in[0,1] \text{ or } \mathbb{R}/\mathbb{Z}}$, define a metric on ΛM by

$$\langle \xi, \eta \rangle := \int_{S^1} \omega_{\gamma(t)} (\xi(t), \eta(t))\, dt$$

for any $\xi, \eta \in T_\gamma \Lambda M$ $\left(\simeq \Gamma(\gamma^* TM) = \left\{ \begin{array}{l} \text{vector fields} \\ \text{along } \gamma(t) \end{array} \right\} \right)$

Then by computation,

$$\underset{\text{w.r.t } \langle , \rangle}{\vec{\nabla}} A_H(\gamma) = J_t (\dot\gamma - X_{H_t}(\gamma)) \Rightarrow \text{the negative flowline } u: \underset{s}{\mathbb{R}} \times \underset{t}{S^1} \to M \text{ is}$$

$$\partial_s u = -\nabla A_H(u(s,t)) = J_t(u(s,t)) \left(\partial_t u - X_{H_t}(u(s,t)) \right) \quad (*)$$

which is a perturbed version of J-hol curve!

Again, one needs to study the moduli space

$$\mathcal{M}(\gamma, \gamma') = \{ u: \mathbb{R}\times S^1 \to M \mid (*) \}$$

☺: experience of J-hol curves can be borrowed

☹: the domain $\mathbb{R}\times S^1$ is non-cpt! ← study asymptotic behavior of such solutions

Assume things worked out, then

$$\partial \gamma = \sum_{\substack{\gamma' \\ \text{ind diff} = 1}} \mathcal{M}(\gamma, \gamma')/_{\mathbb{R}} \cdot \gamma' \quad \text{is well-defined}$$

and $\partial \circ \partial = 0$

$\Rightarrow \quad CF_*(M, \omega, J, H) = \mathbb{Z}_2 \langle \gamma \mid \deg(\gamma) = * \rangle \circlearrowright \partial$

$\Rightarrow \quad HF_*(M, \omega, J, H)$ is well-defined.

<u>Thm</u> (M, ω) symp aspherical, then $HF_*(M, \omega, J, H) \simeq H_*(M; \mathbb{Z}_2)$

$\Rightarrow$ Arnold conj b/c

$$\# \mathrm{Fix}(\varphi_H^1) \geq \# \{\text{closed orbits of } X_H\}^{\text{contractible}} \geq \text{total rank of } HF_*$$
$$= \sum b_i(M; \mathbb{Z}_2)$$

<u>Rmk</u> There are many other versions of Floer homologies, with generators admitting different alg/geo meanings.

<u>Rmk</u> Recall that in Morse theory, the key result is <u>not</u> the well-definedness of Morse homology $HM_*(X, f)$, instead it's the iso: $HM_*(X, f) \simeq H_*(X; \mathbb{Z}_2)$. //

<u>Rmk</u> Working over (M, ω, J) (instead of (M, J)) allows us to define/consider "metric" (on M) and "energy" of a J-hol curve.

2. Contact geometry

(X^{2n-1}, ξ) ξ a hyperplane field and called a (co-oriented) contact structure if $\exists\, \alpha \in \Omega^1(X)$ s.t.

$\operatorname{Ker}\alpha = \xi$ and $\alpha \wedge \underbrace{d\alpha \wedge \cdots d\alpha}_{n-1}$ is a volume form

α is called a contact form

Ex. $(\mathbb{R}^{2n-1}, \xi_{std})$ $\xi_{std} = \operatorname{Ker}\left(dz - \sum_{i=1}^{n-1} y_i dx_i\right)$

$x_1, y_1, \ldots, x_{n-1}, y_{n-1}, z$

Rmk All are encouraged to draw $(\mathbb{R}^3, \xi_{std})$ by hand.

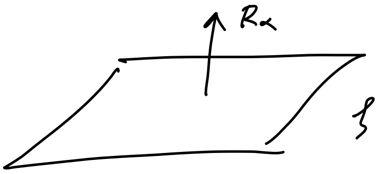
Rmk The data of contact forms is auxiliary. If α is a contact form of ξ, then $e^f \cdot \alpha$ is also a contact form of ξ.

Any contact mfd (X, ξ) admits a dyn system itself: fix α s.t. $\xi = \ker\alpha$, one can solve a vector field R_α from

$$d\alpha(R_\alpha, -) = 0 \quad \text{and} \quad \alpha(R_\alpha) = 1.$$

$d\alpha|_\xi$ is non-deg.

in a unique way. It is called the Reeb vector field of α.



$\varphi^t_{R_\alpha}$ the Reeb flow (w.r.t. α)

Again, one should be interested in closed orbits of $\varphi^t_{R_\alpha}$.

<u>Conjecture (Weinstein)</u> For any closed contact mfd (X, ξ), for any fixed α s.t $\xi = \ker\alpha$, there $\exists$ at least 1 closed orbit of $\varphi^t_{R_\alpha}$.

<u>Rmk</u>. Different from Arnold conj, we do not restrict to time-one.

<u>Rmk</u>. Conclusion is wrong if (X, ξ) is non-cpt: R_α in $(\mathbb{R}^3, \xi_{std})$ is ∂_z, so $\varphi^t_{R_\alpha}(z) = z + t$.

<u>Rmk</u> In dim 3, this has been proved by Taubes.
Other dims have individual results, but in general it is contraversial.

//

Ex. (M^4, ω) symp mfd. $H: M \to \mathbb{R}$ autonomous Ham fcn.

Consider the level set $H^{-1}(c) \subset M$, where c is regular.

- $H^{-1}(c)$ is a mfd of dim 3.
- flow lines of X_H stay inside $H^{-1}(c)$ $\leftarrow$ $dH(X_H) = -\omega(X_H, X_H) = 0$

Suppose near $H^{-1}(c)$, $\exists$ a vector field $Y \pitchfork H^{-1}(c)$ and $\underbrace{\mathcal{L}_Y \omega = \omega}_{(*)}$

$$(*) \iff d\underbrace{\iota_Y\omega}_{=:\lambda} + \cancel{\iota_Y d\omega}^{0} = \omega$$

$$\iff \exists \text{ 1-form } \underset{\substack{\uparrow \\ \text{called a Liouville form}}}{\lambda} \text{ (near } H^{-1}(c)) \text{ s.t. } \omega = d\lambda.$$

Then one can check that $\Big(H^{-1}(c), \underbrace{\xi = \ker(\lambda|_{H^{-1}(c)})}_{\text{dim 2 hyperplane field}}\Big)$ is a contact mfd. Moreover,

$$R_\lambda = \frac{X_H}{dH(Y)} \leftarrow \text{non-zero b/c } Y \pitchfork H^{-1}(c)$$

Verify:
$$\begin{aligned}
\lambda(R_\lambda) &= (\iota_Y \omega)\left(\frac{X_H}{-dH(Y)}\right) \quad \text{no minus sign} \\
&= \omega\left(Y, \frac{X_H}{dH(Y)}\right) \\
&= -\frac{1}{dH(Y)}\, \omega(X_H, Y) = 1.
\end{aligned}$$

$$\iota_{X_H} \underset{\substack{\uparrow \\ \lambda|_{H^{-1}(c)}}}{d\lambda} = \iota_{X_H} \omega|_{H^{-1}(c)} = \omega|_{H^{-1}(c)}(X_H, -) = -dH(\underset{\substack{\uparrow \\ \text{input from } H^{-1}(c)}}{-}) = 0$$

Then Taube's result implies that $\exists$ a closed orbit (closed characteristic) of the Ham flow X_H (not nec at $t=1$) on $H^{-1}(c)$!

<u>Rmk</u> Weinstein, Rabinowitz proved this $\exists$-result in 1978. for any dim. (Of course, not every cpt contact str can be viewed as the level set of a Ham system).

Example above gives a hint that symp geo is related with contact geo in a natural way.

Def A Liouville domain is a symp mfd with boundary (W, ω) s.t. $\exists\, Y$ a v.f. <u>on W</u>, $\pitchfork \partial W$, and $\underset{\substack{\uparrow \\ \omega \text{ exponentially grows} \\ \text{or shrinks.}}}{\underline{\mathcal{L}_Y \omega = \omega}}$.

By discussion above, $(\partial W,\ \xi = \ker(\iota_Y \omega)|_{\partial W})$ is a contact mfd.

Ex. $\underset{\substack{\cap \\ (\mathbb{R}^{2n},\, \omega_{std})}}{(W, \omega)} = (\{\sum x_i^2 + y_i^2 \leq 1\},\ \omega_{std}|_{\{\dots\}})$ is a Liouville domain

and $Y = \frac{1}{2} \sum_{i=1}^{n} (x_i \partial_{x_i} + y_i \partial_{y_i})$ radial vector field.

$\Rightarrow (\partial W, \xi) = (S^{2n-1}, \xi_{std})$ contact mfd with the standard contact str.

observation: $\lambda = \iota_Y \omega$ is fixed but $\lambda|_{\partial W}$ varies, depending on ∂W.

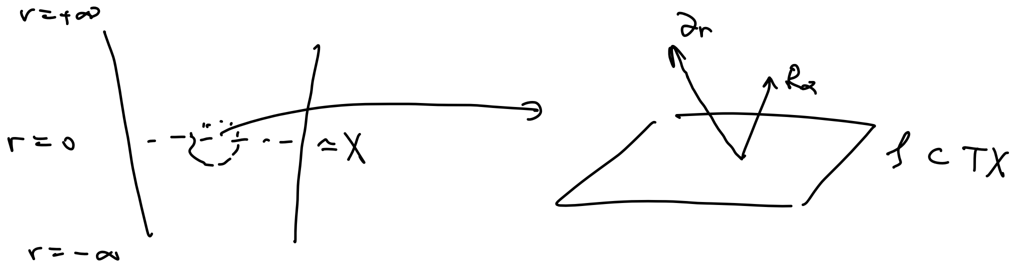
(S^{2n-1}, ξ_{std}) $\qquad$ (S_t, ξ_{std})

One can also fix S^{2n-1} but deform ξ_{std} to ξ_t (through contact str), via deforming Y, then Gray's stability thm (Thm 1.6 in [Wen]) shows that $\exists\, \underset{\substack{\uparrow \\ \text{contactomorphism}}}{\varphi_t} : S^{2n-1} \circlearrowleft$ s.t. $(\varphi_t)_* \xi_t = \xi_{std}$.

//

From contact mfd (X,ξ), one can build a symp mfd via "symplectization": fix α s.t. $\xi = \ker\alpha$, then

$$M := \underset{r}{\mathbb{R}} \times X \quad \text{and} \quad \omega = d(e^r\alpha) \qquad e^r dr \wedge \alpha$$



Rmk Sometimes papers use another convention $M = [0,\infty)\times X$ and $\omega = dr\wedge\alpha$ and $X \simeq \{1\}\times X$.

How does a J-hol curve look like in a symplectization?

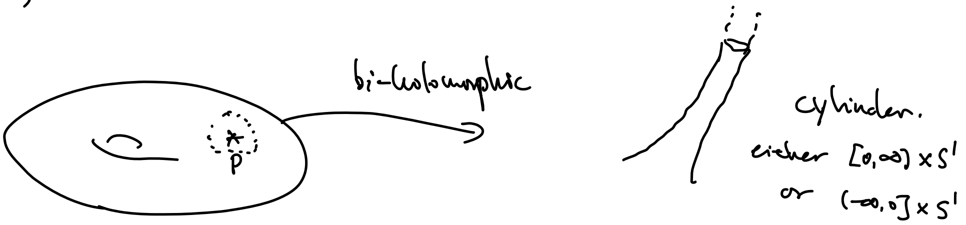
Pick J on $\mathbb{R}\times X$ s.t.

- J is invariant under translation in r, i.e. $J_{(a,x)} = J_{(a+r,x)}$
- $J(\partial_r) = R_\alpha$ and $J(R_\alpha) = -\partial_r$
- Restricted at ξ, $\omega(\cdot, J\cdot)$ is a metric (on bundle $\begin{matrix}\xi\\ \downarrow\\ X\end{matrix}$)

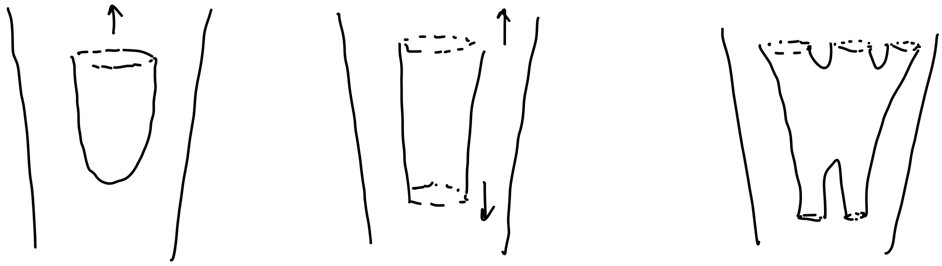
Prop $u: (\Sigma, j) \to (\mathbb{R}\times X, \omega)$ by $(u_{\mathbb{R}}, u_X)$. if u is J-hol, then $u_{\mathbb{R}}$ is a subharmonic function ($\Delta u_{\mathbb{R}} \geq 0$)

$\Rightarrow$ Σ cannot be compact.

Modify Σ to be punctured $\dot{\Sigma} = \Sigma \setminus \{p_1, \cdots, p_m\}$.



Ex



$\Sigma \setminus \{p\} \simeq \mathbb{D}^2$ $\qquad$ $\Sigma \setminus \{p, q\} \simeq$ cylinder $\qquad$ $\Sigma \setminus \{p_1, \cdots, p_3\}$

$\lim_{s \to \infty} u(s,t) = \gamma(t)$ positive (asymptotic) end

$\lim_{s \to -\infty} u(s,t) = \gamma(t)$ negative (asymptotic) end

Thm. Under a finite energy condition, $\gamma(t)$ is a closed Reeb orbit.
(later)

Rmk Recall that in (Ham) Floer homology, ∂ is also involving the study of asymptotic behavior when $s \to \pm\infty$.

Rmk Similarly to symp geo situation, studying the moduli space of punctured $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is also crucial.

need to add asymp. cond.

3. Symplectic embedding

Given two Liouville domain (U, ω_U) and (V, ω_V), a symp emb is an emb $\varphi: U \hookrightarrow V$ s.t. $\varphi^*\omega_V = \omega_U$.

Ex. $(\mathbb{R}^{2n}, \omega_{std})$ $(\simeq (\mathbb{C}^n, \omega_{std}))$

$$E(a_1, \dots, a_n) = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n \;\middle|\; \frac{\pi|z_1|^2}{a_1} + \dots + \frac{\pi|z_n|^2}{a_n} \le 1 \right\}$$ ← symplectic ellipsoid

$$B^{2n}(r) = E(r, \dots, r), \quad Z^{2n}(R) = E(R, \underbrace{\infty, \dots, \infty}_{\text{no constraint on } z_2, \dots, z_n})$$

$B^{2n}(r)$: symplectic ball, $Z^{2n}(R)$: symplectic cylinder.

$$P(a_1, \dots, a_n) = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n \;\middle|\; \pi|z_1|^2 \le a_1, \dots, \pi|z_n|^2 \le a_n \right\}$$ ← polydisk

Rmk. $\partial P(a_1, \dots, a_n)$ is not smooth.

Rmk. All cases above admit $\mathbb{T}^n$-action by $(\theta_1, \dots, \theta_n) \cdot (z_1, \dots, z_n)$ defined by $(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$. $\Longrightarrow$ toric domain

Finding obstructions of embedding is a central topic in symp. geo.