$\Rightarrow \partial^2 = 0$

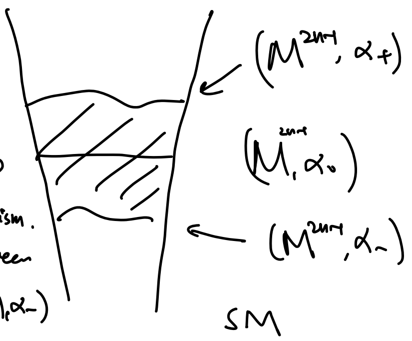
$\Rightarrow \quad HC^h_*(M, \alpha, J) := H_*(CC^h_*(M,\alpha), \partial).$

<u>Question:</u> Dependence on α and J?

Given $\alpha_I = f_I\alpha_0$ of ξ, and J by rescaling α_+ by constant $c \gg 1$,

assume $f_+ > f_-$ pointwise:

this forms a symp corbodism W between (M,α_+) and (M,α_-)

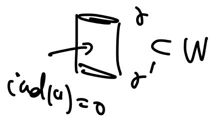


$HC^h_*(M, c\alpha_+, J_c)$ ← modification from J

$\simeq HC^h_*(M, \alpha_+, J)$

Then consider $\Phi^J_*, \; CC^h_*(M,\alpha_+) \longrightarrow CC^h_*(M,\alpha_-)$

$\gamma \longrightarrow \gamma'$



and verify that

$$\begin{tikzcd}
CC^h_*(M,\alpha_+) \arrow[r, "\Phi^J_*"] \arrow[d, "\partial_+"'] \arrow[dr, phantom, "\circlearrowleft"] & CC^h_*(M,\alpha_-) \arrow[d, "\partial_-"] \\
CC^h_{*-1}(M,\alpha_+) \arrow[r, "\Phi^J_{*-1}"] & CC^h_{*-1}(M,\alpha_-)
\end{tikzcd}$$

this diagram commutes, so Φ^J_* is a chain map.

$\Rightarrow \quad \Phi^J_* : HC^h_*(M, \alpha_+, J) \longrightarrow HC^h_*(M, \alpha_-, J)$

a well-defined homomorphism.

In a similar way, for J, J', $\exists\ \Phi^{J,J'}_* : HC^h_*(M, \alpha, J) \to HC^h_*(M, \alpha, J')$ ← any α

$$\Rightarrow \quad \begin{tikzcd} HC^h_*(M, \alpha_+, J) \arrow[r, "\Phi^J_*"] \arrow[rr, bend right, "\Phi"'] & HC^h_*(M, \alpha_-, J) \arrow[r, "\Phi^{J,J'}_*"] & HC^h_*(M, \alpha_-, J') \end{tikzcd}$$

Then switching (α_+, J) and (α_-, J'), we get Ψ and verify $\Psi \cdot \Phi = \Phi \cdot \Psi = \mathbb{1}$ ← counting trivial cylinders.

$\Rightarrow \quad HC^h_*(M, \alpha, J) \underset{\text{def}}{=} HC^h_*(M, \xi)$.

this is called the cylindrical contact homology of contact mfd (M, ξ).

Prop: $\varphi : (M, \xi) \to (M', \xi')$ contactomorphism s.t. $\varphi_* h = h'$ for $h \in [S^1, M]$. Assume $\exists$ contact 1-form α' of ξ' s.t. α' is h'-admissible, then $\exists$ contact 1-form α of ξ s.t. α is h-admissible and $HC^h_*(M, \xi) = HC^{h'}_*(M', \xi')$.

$\Rightarrow \quad HC^h_*(M, \xi)$ defines a contact invariant (of contact str) up to contactomorphism.

One can use $HC^h_*(M, \xi)$ to distinguish different contact strs.

Ex. $\mathbb{T}^3 = S^1 \times S^1 \times S^1$, where $S^1 = \mathbb{R}/\mathbb{Z}$, in coordinate (ρ, φ, θ).

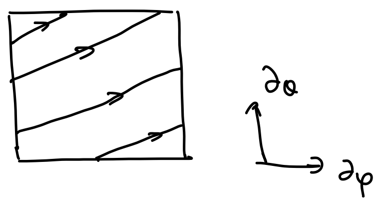
Consider $\alpha_k := \cos(2\pi k\rho)\,d\theta + \sin(2\pi k\rho)\,d\varphi$ $\quad k\in\mathbb{N}_{\geq 1}$.

Then
$$\alpha_k\wedge d\alpha_k = \big(\cos(2\pi k\rho)\,d\theta + \sin(2\pi k\rho)\,d\varphi\big)\wedge$$
$$2\pi k\rho\big(-\sin(2\pi k\rho)\,d\rho\wedge d\theta + \cos(2\pi k\rho)\,d\rho\wedge d\varphi\big)$$
$$= 2\pi k\rho\big(-\sin^2(2\pi k\rho)\,d\varphi\wedge d\rho\wedge d\theta + \cos^2(2\pi k\rho)\,d\theta\wedge d\rho\wedge d\varphi\big)$$
$$= 2\pi k\rho\,d\rho\wedge d\varphi\wedge d\theta > 0$$

Reeb vector field R_{α_k} is

$$R_{\alpha_k} = \sin(2\pi k\rho)\,\partial_\varphi + \cos(2\pi k\rho)\,\partial_\theta$$

$\Rightarrow$ closed Reeb orbits are in the form:

$$\{\rho\}\times\mathbb{T}^2 \;\simeq\; \{\rho\}\times$$


$$\Big(\mathbb{T}^3 = \coprod_{\rho\in\mathbb{R}/\mathbb{Z}}\{\rho\}\times\mathbb{T}^2\Big)$$

In particular, all closed Reeb orbits are non-contractible.

<u>Rmk</u> These orbits are degenerate (so extra work is needed).

<u>Thm</u> For $h = [t\to(0,0,t)]$, we have

$$HC^h_*(\mathbb{T}^3,\xi_k) = \begin{cases} \mathbb{Z}_2^k & * \text{ is odd} \\ \mathbb{Z}_2^k & * \text{ is even}\end{cases}$$

$\Rightarrow$ For different k, l, $\xi_k \overset{\text{cont}}{\not\cong} \xi_l$.

(Here one needs to practice an Exercise: if $h, h' \in [S^1, T^3]$, primitive and mapped to trivial class under projection $T^3 \to S^1$, $(\rho, \varphi, \theta) \to \rho$. Then $\exists$ a contactomorphism $\varphi : (T^3, \xi_k) \circlearrowleft$ s.t. $\varphi_* h = h'$.)

Rmk Observe that for each $k \in \mathbb{N}_{\geq 1}$, consider smooth deformations

$$\ker\big((1-s)\alpha_k + s\, d\rho\big) \qquad s \in [0,1]$$

and we get from ξ_k $(= \ker \alpha_k)$ to $\ker(d\rho)$. ← not contact. So topologically all hypersurfaces ξ_k (for $k \in \mathbb{N}_{\geq 1}$) are the same.

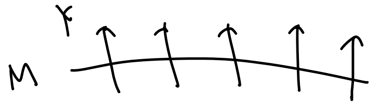
($\Rightarrow$ we need refined contact geo machinery to distinguish ξ_k, ξ_l).

Rmk This result has been known since 90s (by Giroux, Kanda).

2. Neck-stretching

(X^{2n}, ω) closed symp mfd

$M \subset X^{2n}$ hypersurface of contact type ← or more general stable hypersurface (with induced stable Ham str on M).



$\lambda = \iota_Y \omega$

$\Rightarrow$ a NBH of M in X can be identified with $\Big(\underbrace{(-\varepsilon, \varepsilon)_r \times M}_{\text{a part of symplectization}},\; r\, d\lambda\Big)$

For any $k \in \mathbb{N}$, consider a reparametrization

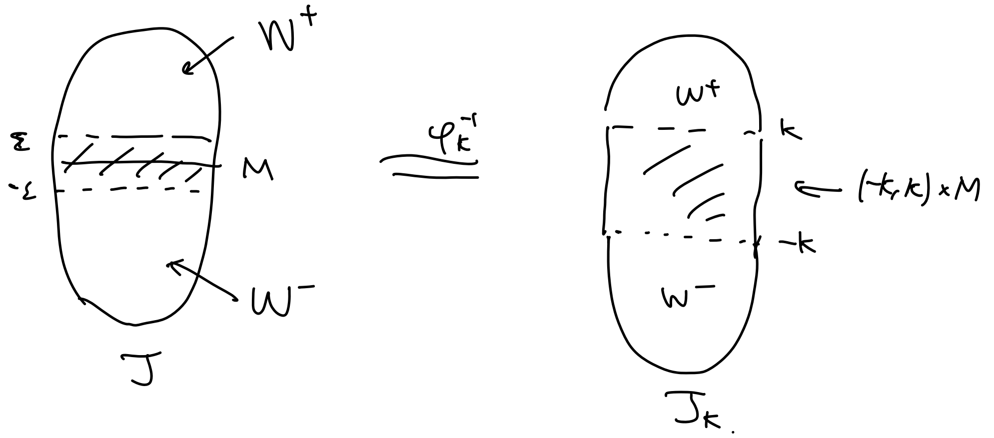
$$(-k, k) \longrightarrow (-\varepsilon, \varepsilon)$$

s.t. $(-k, k) \times M \xrightarrow[\varphi_k]{\text{symp}} (-\varepsilon, \varepsilon) \times M$

Then if we fix a compatible a.c.s $\underbrace{J \text{ on } (-\varepsilon, \varepsilon) \times M}_{\text{can be induced by a compatible a.c.s } J \text{ on } (X^{2n}, \omega)}$,

then $(\varphi_k^{-1})_* J =: J_k$ is a compatible a.c.s on $(-k, k) \times M$

and extend to J_k on (X^{2n}, ω).



As $k \to +\infty$, up to symp, the ambient space does not change (X^{2n}, ω)

but a.c.s J has been "stretched" around M ⟵ called neck-stretching.

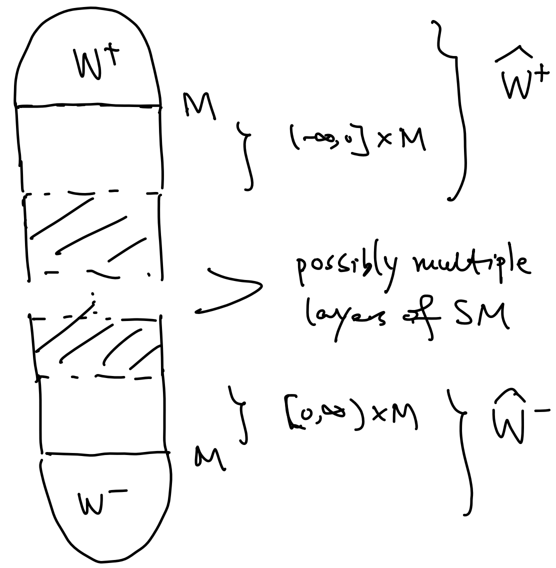
Now, take a J-hol curve $u: (S^2, j) \longrightarrow (X^{2n}, \omega, J)$ and carry out this neck-streching:

$$u \rightsquigarrow u_k: (S^2, j) \longrightarrow (X^{2n}_k, \omega, J_k) \quad J_k\text{-hol}$$

and in the limit, we will see a "hol building"

(as an intermediate type between Gromov compactness and SFT compactness)

U_∞: bubble tree $\longrightarrow$



Application: obstruct embedding of a Lag submfd $\hookrightarrow (X^{2n},\omega)$

(b/c "Lag" Weinstein Thm says any Lag submfd $L \subset (X^{2n},\omega)$ admits a NBH $\mathcal{U}\,(\subset X^{2n})$ s.t. $\mathcal{U} \overset{\Phi}{\cong} \underbrace{\text{NBH of } 0_L \subset T^*L}$.

Then equip L with a metric g, then $\overset{\Downarrow}{U_g^*L} \subset T^*L$

and $\partial U_g^*L \,(= S_g^*L)$ unit cosphere bundle will serve as a hypersurface $M \subset (X^{2n},\omega)$ under the symplectomorphism Φ)

3. Obstructing domain embeddings

$$L(r,s) = \underset{\mathbb{C}}{\underset{\cap}{S^1(r)}} \times \underset{\mathbb{C}}{\underset{\cap}{S^1(s)}} \subset \mathbb{C}^2 \quad \text{split Lag torus.}$$

<u>Question</u> Given a domain $U \subset \mathbb{C}^2$, which (r,s) s.t. $\underset{\overset{\shortparallel}{T^2}}{L(r,s)} \hookrightarrow U$?

e.g. $U = E(a,b)$.

<u>Rmk</u> One can assume $r \leq s$.

<u>Thm</u> (Hind - Z. 2020)

For $E(a,b)$ with $\frac{b}{a} \in \mathbb{N}_{\geq 2}$, we know

$$\underset{(x \geq 1)}{L(1,x)} \hookrightarrow E(a,b) \iff \begin{cases} x < b\left(1-\frac{1}{a}\right) & \text{if } 1 \leq x < 2 \\ \text{either } a > 2 \\ \text{or } x < b\left(1-\frac{1}{a}\right) & \text{if } x \geq 2 \end{cases}$$

"$\Longrightarrow$" obstructive part (given by SFT)

"$\Longleftarrow$" Construction (flexibility)

(Then $L(r,s) \hookrightarrow E(a,b) \iff L\left(1,\frac{s}{r}\right) \hookrightarrow E\left(\frac{a}{r}, \frac{b}{r}\right)$

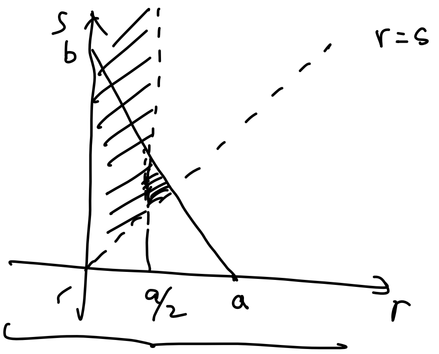
$$\iff \begin{cases} \frac{s}{r} < \frac{b}{r}\left(1-\frac{a}{r}\right) & \text{if } 1 \leq \frac{s}{r} < 2 \\ \text{either } \frac{a}{r} > 2 \\ \text{or } \frac{s}{r} < \frac{b}{r}\left(1-\frac{a}{r}\right) & \text{if } \frac{s}{r} \geq 2 \end{cases}$$

$\frac{a}{r} + \frac{b}{s} < 1$

$$\iff \begin{cases} s < b\left(1-\frac{a}{r}\right) & \text{if } r \leq s < 2r \\ \text{either } \frac{a}{2} > r \leftarrow \text{no condition on } s \\ \text{or } s < b\left(1-\frac{a}{r}\right) & \text{if } s \geq 2r \end{cases}$$

Encode (r,s) (with $r \leq s$) into a 2-dim picture:

$L(r,s) \hookrightarrow E(a,b) \iff$



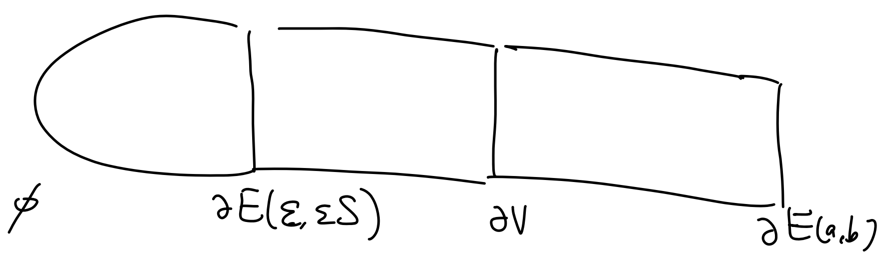
(called the Ham shape invariant of $E(a,b)$.)

About the proof of "$\Longrightarrow$": if $L(1,x) \hookrightarrow E(a,b)$
then NBH V (of image of $L(1,x)$) $\hookrightarrow E(a,b)$

Consider $E(\varepsilon, \varepsilon S) \hookrightarrow V$ when $0 < \varepsilon \ll 1$ sufficiently small.

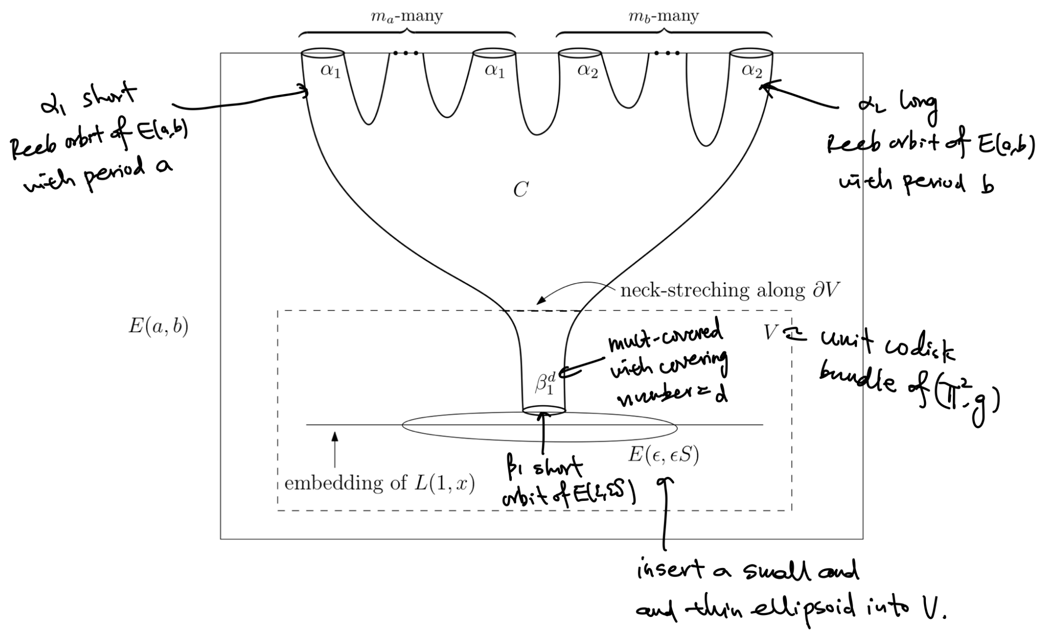
Then $E(\varepsilon, \varepsilon S) \hookrightarrow V \hookrightarrow E(a,b)$ induces the following cobordisms:

$$\underbrace{\emptyset \longrightarrow E(\varepsilon,\varepsilon S)}_{E(\varepsilon,\varepsilon S)} \underbrace{\longrightarrow V}_{V \setminus \mathrm{im}(E(\varepsilon,\varepsilon S))} \underbrace{\longrightarrow E(a,b)}_{E(a,b)\setminus \mathrm{im}(V)}$$

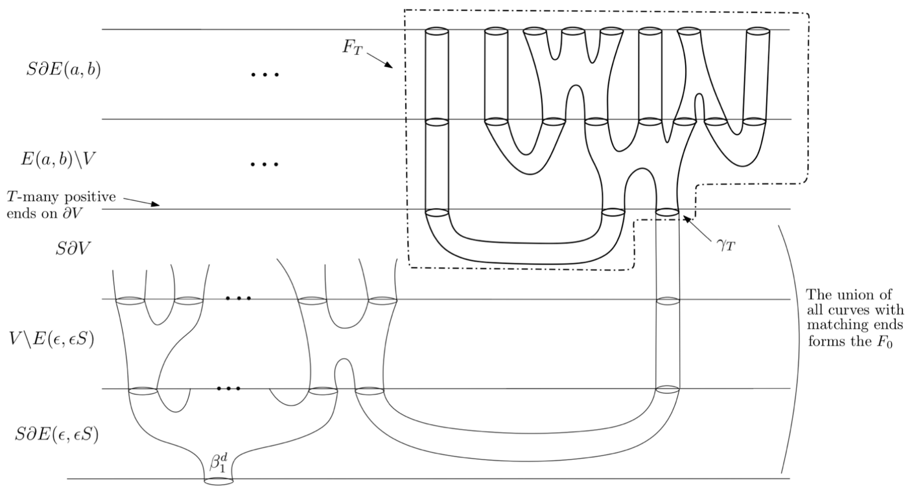


Consider a J-hol curve in the following type:

e.g. related to $L(1,x) \hookrightarrow E(a,b)$



After neck-stretching, one could see the limit curve: (holomorphic building)



Then the proof starts from analysis on different components... //

Thank You !