Application of SFT

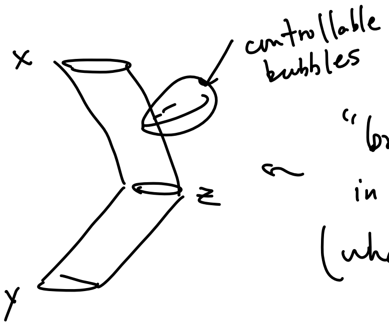
1. Cylindrical contact homology

Similarly to Ham Floer homology, given a contact mfd $(M, \xi = \ker\alpha)$ one can consider SM symplectization and a Floer theory based on closed Reeb orbits, connected by J-hol cylinders

← J-hol cylinder (with index = 1)

Question: define ∂ and show $\partial^2 = 0$.

An obvious trouble: $\partial^2 = 0$ involves counting index = 2 cylinder and in particular "boundary" elements in compactification of moduli space.

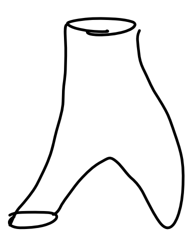


$\sim$ "broken" flowline in Ham Floer homology (where target in cpt)

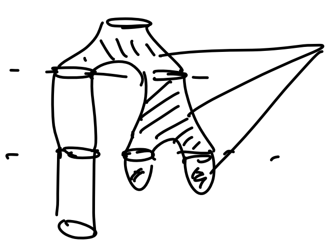
$\Longrightarrow$ $\partial^2 = 0$

$\Updownarrow$

{"broken" flowlines} = b/d of cpt 1-dim mfd.



$\xrightarrow[\text{limit}]{\text{SFT}}$



a hol building

these components are in general not in the top type of cylinders (due to SM is non-cpt).

Here is a quick way to fix this $\rightsquigarrow$ simply requiring there does NOT exist any contractible closed Reeb orbits.

Def. Fix a homotopy class $h \in [S^1, M]$ and assume it is primitive (i.e. not iteration of another htp class).

Denote $P_h(\alpha) = \{$ closed Reeb orbits of $(M, \xi = \ker\alpha)$ in class $h\}$

We call α for contact str ξ h-admissible if (1) $\nexists$ any contractible closed Reeb orbit; (2) any element in $P_h(\alpha)$ is non-deg.

$\rightarrow$ to guarantee that limit behavior of J-hol curve in SM is "uniform".

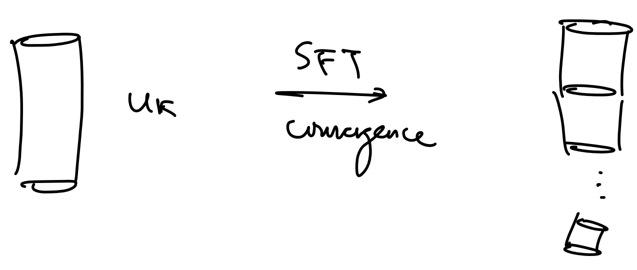
Rmk "h is primitive" is to make sure no multiple covering.

$\Longrightarrow$ consider only hol cylinders between closed Reeb orbit (from $P_h(\alpha)$) with one positive end and one negative end, then

$$\left\{ J \text{ a.c.s on } SM \;\middle|\; \begin{array}{c}\forall \text{ J-hol cylinder induces} \\ \text{Fredholm operator}\end{array} \right\}$$

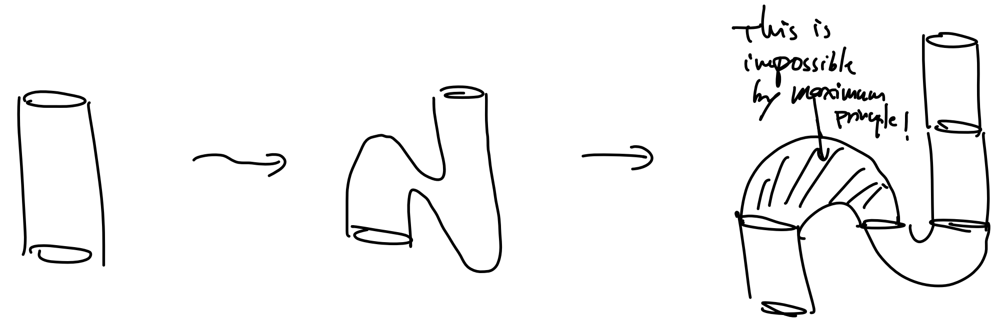
is generic (b/c no multiple cover $\Longleftrightarrow$ simple $\Longrightarrow$ somewhere injective).

No contractible closed Reeb orbits $\Longrightarrow$ a compactness result (Prop 10.19 in [Wen])



hol building where each level is a cylinder.

e.g.



$\Rightarrow$ Under this h-admissible condition, we can use cylinders to form a Floer theory.

Define $\quad CC^h_*(M^{2n-1}, \alpha) := \bigoplus_{\gamma \in P_h(\alpha)} \mathbb{Z}_2 \langle \gamma \rangle \quad \longleftarrow$ each admits a grading $n-3+\mu_{CZ}^{\tau}(\gamma) \in \mathbb{Z}$ (depending on a choice of trivialization)

$$\partial \gamma = \sum_{\gamma' \in P_h(\alpha)} \#_{\mathbb{Z}_2} \mathcal{M}^{\text{index}=1}\left(\boxed{u}\,{}^{\gamma}_{\gamma'} \right) \Big/ \mathbb{R} \cdot \gamma'$$

Here, index $=$ ind(u) ($=$ difference of grading of γ and γ')

(Note that a trivial cylinder $\square^{\gamma}_{\gamma}$ is identified with a point after modulo $\mathbb{R}$-translation.)

Prop 10.21 $\Rightarrow$ For a generic J, fixed $\gamma^{\pm} \in P_h(\alpha)$,

$$\overline{\mathcal{M}^{\text{ind}=2}\left(\square^{\gamma_+}_{\gamma_-} \right)} = \text{cpt 1-dim mfd with b/d}$$

and $\partial\, \overline{\mathcal{M}^{\text{ind}=2}\left(\square^{\gamma_+}_{\gamma_-} \right)} = \coprod_{\gamma_0 \in P_h(\alpha)} \mathcal{M}^{\text{ind}=1}\left(\square^{\gamma_+}_{\gamma_0} \right) \times \mathcal{M}^{\text{ind}=1}\left(\square^{\gamma_0}_{\gamma_-} \right).$

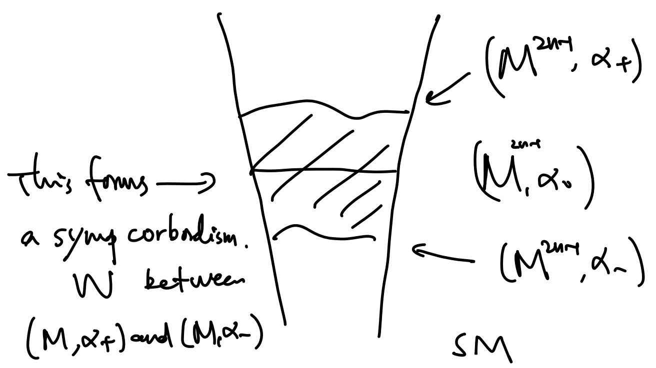
$\Rightarrow \partial^2 = 0$

$\Rightarrow \quad HC^h_*(M, \alpha, J) := H_*(CC^h_*(M,\alpha), \partial).$

<u>Question:</u> Dependence on α and J?

Given $\alpha_I \overset{= f_I \alpha_0}{}$ of ξ, and J by rescaling α_+ by constant $c \gg 1$,

assume $f_+ > f_-$ pointwise:

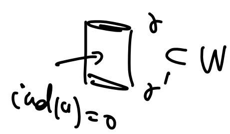


$HC^h_*(M, c\alpha_+, J_c)$ ← modification from J

$\simeq HC^h_*(M, \alpha_+, J)$

Then consider $\Phi^J_*, \; CC^h_*(M,\alpha_+) \longrightarrow CC^h_*(M,\alpha_-)$

$\gamma \longrightarrow \gamma'$



and verify that

$$\begin{tikzcd}
CC^h_*(M,\alpha_+) \arrow[r, "\Phi^J_*"] \arrow[d, "\partial_+"'] \arrow[dr, phantom, "\circlearrowleft"] & CC^h_*(M,\alpha_-) \arrow[d, "\partial_-"] \\
CC^h_{*-1}(M,\alpha_+) \arrow[r, "\Phi^J_{*-1}"] & CC^h_{*-1}(M,\alpha_-)
\end{tikzcd}$$

this diagram commutes, so Φ^J_* is a chain map.

$\Rightarrow \quad \Phi^J_* : HC^h_*(M, \alpha_+, J) \longrightarrow HC^h_*(M, \alpha_-, J)$

a well-defined homomorphism.

In a similar way, for J, J', $\exists\ \Phi^{J,J'}_* : HC^h_*(M, \overset{\text{any } \alpha}{\alpha}, J) \to HC^h_*(M, \alpha, J')$

$$\Rightarrow \quad \begin{tikzcd} HC^h_*(M, \alpha_+, J) \arrow[r, "\Phi^J_*"] \arrow[rr, bend right, "\Phi"'] & HC^h_*(M, \alpha_- , J) \arrow[r, "\Phi^{J,J'}_*"] & HC^h_*(M, \alpha_-, J') \end{tikzcd}$$

Then switching (α_+, J) and (α_-, J'), we get Ψ and verify

$\Psi \circ \Phi = \Phi \circ \Psi = \mathbb{1} \leftarrow$ counting trivial cylinders.

$\Rightarrow \quad HC^h_*(M, \alpha, J) \underset{\text{def}}{=} HC^h_*(M, \xi)$.

this is called the cylindrical contact homology of contact mfd (M, ξ).

Prop: $\varphi : (M, \xi) \to (M', \xi')$ contactomorphism s.t. $\varphi_* h = h'$ for $h \in [S^1, M]$. Assume $\exists$ contact 1-form α' of ξ' s.t. α' is h'-admissible, then $\exists$ contact 1-form α of ξ s.t. α is h-admissible and $HC^h_*(M, \xi) = HC^{h'}_*(M', \xi')$.

$\Rightarrow \quad HC^h_*(M, \xi)$ defines a contact invariant (of contact str) up to contactomorphism.

One can use $HC^h_*(M, \xi)$ to distinguish different contact strs.

Ex. $\mathbb{T}^3 = S^1 \times S^1 \times S^1$, where $S^1 = \mathbb{R}/\mathbb{Z}$, in coordinate (ρ, φ, θ).

Consider $\alpha_k := \cos(2\pi k\rho)\, d\theta + \sin(2\pi k\rho)\, d\varphi$. $\quad k \in \mathbb{N}_{\geq 1}$.

Then $\alpha_k \wedge d\alpha_k = \left(\cos(2\pi k\rho) d\theta + \sin(2\pi k\rho) d\varphi\right) \wedge$

$$2\pi k\rho\left(-\sin(2\pi k\rho)\, d\rho\wedge d\theta + \cos(2\pi k\rho)\, d\rho \wedge d\varphi\right)$$

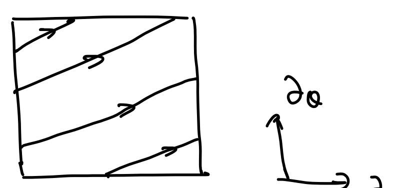
$$= 2\pi k\rho\left(-\sin^2(2\pi k\rho)\, d\varphi\wedge d\rho\wedge d\theta + \cos^2(2\pi k\rho)\, d\theta\wedge d\rho\wedge d\varphi\right)$$

$$= 2\pi k\rho\; d\rho\wedge d\varphi\wedge d\theta > 0$$

Reeb vector field R_{α_k} is

$$R_{\alpha_k} = \sin(2\pi k\rho)\,\partial_\varphi + \cos(2\pi k\rho)\,\partial_\theta$$

$\Rightarrow$ closed Reeb orbits are in the form:

$\{\rho\}\times \mathbb{T}^2 \simeq \{\rho\}\times$ 

$\left(\mathbb{T}^3 = \coprod_{\rho\in\mathbb{R}/\mathbb{Z}} \{\rho\}\times \mathbb{T}^2\right)$

In particular, all closed Reeb orbits are non-contractible.

<u>Rmk</u> These orbits are degenerate (so extra work is needed).

<u>Thm</u> For $h = [t \to (0,0,t)]$, we have

$$HC_*^h(\mathbb{T}^3, \xi_k) = \begin{cases} \mathbb{Z}_2^k & * \text{ is odd} \\ \mathbb{Z}_2^k & * \text{ is even}\end{cases}$$

$\Longrightarrow$ For different k, l, $\xi_k \overset{\text{cont}}{\not\cong} \xi_l$.

(Here one needs to practice an Exercise: if $h, h' \in [S^1, \mathbb{T}^3]$, primitive and mapped to trivial class under projection $\mathbb{T}^3 \to S^1$, $(\rho, \varphi, \theta) \to \rho$. Then $\exists$ a contactomorphism $\varphi: (\mathbb{T}^3, \xi_k) \circlearrowleft$ s.t. $\varphi_* h = h'$.)

Rmk Observe that for each $k \in \mathbb{N}_{\geq 1}$, consider smooth deformations

$$\ker((1-s)\alpha_k + s\, d\rho) \qquad s \in [0,1]$$

and we get from ξ_k ($= \ker \alpha_k$) to $\ker(d\rho)$. not contact So topologically all hypersurfaces ξ_k (for $k \in \mathbb{N}_{\geq 1}$) are the same.

($\Rightarrow$ we need refined contact geo machinery to distinguish ξ_k, ξ_l).

Rmk This result has been known since 90s (by Giroux, Kanda).