PREPARATION SHEET (MATH7431P - USTC, FALL 2025)

This document outlines the source of questions for the upcoming Final Exam. All exam questions will be selected from the following question pool, which consists of: key concepts and statements from lecture notes, problems from Homework One through Homework Four, and proofs of propositions/theorems presented in the notes. The pool, and consequently the exam, contains three categories of problems:

A : Problems on Definitions and Statements;
B : Problems on Computations;
C : Problems on Proofs.

For Category A: **10 problems (each worths 5 points)** will be selected for the exam from a pool of 20 problems. For Category B: **2 problems (each worths 10 points)** will be selected from a pool of 5 problems. For Category C: **2 problems (each worths 15 points)** will be selected from a pool of 3 problems.

A. PROBLEMS on DEFINITIONS and STATEMENTS.

(1) Bundle-valued forms (definition and its construction)

(2) Connections on a vector bundle (definition and basic properties).

(3) Sobolev embedding theorems (statements and definitions of involving notations).

(4) Convolution and mollifier (definition and basic properties).

(5) J-holomorphic curve (definition) and its regularity theorem.

(6) Description of $\bar{\partial}_J$ in terms of a section of a certain bundle.

(7) Fredholm operator between Banach spaces (definition and basic properties).

(8) Implicit function theorem in the setting of Banach manifolds.

(9) Carleman Similarity Principle (statement).

(10) Gromov compactness theorem (statement, with target being a closed symplectic manifold and domain being Riemann surfaces without marked points).

(11) Spectral flow (definition, in the finite-dimensional setting).

(12) Stably framed Hamiltonian structure (definition) on an odd-dimensional manifold and its symplectization.

(13) Definition of a J-holomorphic being simple (in terms of the covering theorem for J-holomorphic curves), not only using the term "somewhere injective".

(14) Definition of the first Chern number of complex vector bundles over a closed Riemann surface, and dimension formula of the moduli space of simple curves $\mathcal{M}^*_{J,A}$ (where target is a closed almost complex structure (M, J) and domain is (Σ, j), with respect to a fixed class $A \in H_*(M; \mathbb{Z})$).

(15) Definition of a pointed Riemann surface (Σ_g, j, Θ) and definition of being stable, and list all possible non-stable cases.

(16) Definitions of energy in different settings: J-holomorphic curve into closed symplectic manifold, J-holomorphic curve into a symplectization, Floer trajectory in Hamiltonian Floer homology theory.

(17) State the convergence phenomenon in a rigorous way when the limit of a sequence of gradient flowlines is a "broken flowline" in the Morse setting.

(18) Removal of singularities (statement).

(19) Definition of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,\ell}$.

(20) Draw a picture of a holomorphic building of height $1|1|3$ with arithmetic genus 2 and 3 marked points, and the number of connected components of domain is 13.

B. PROBLEMS on COMPUTATIONS.

(1) HW One, Exercise 5.

(2) HW Two, Exercise 1.

(3) HW Two, Exercise 4.

(4) HW Three, Exercise 3.

(5) HW Four, Exercise 1.

C. PROBLEMS on PROOFS.

(1) Fix a closed symplectic manifold (M, ω) and an ω-compatible almost complex structure J. Prove that there exist a constant $\hbar > 0$ and $C > 0$ such that for any

J-holomorphic curve $u : (\Sigma, j) \to (M, \omega, J)$, where (Σ, j) is a Riemann surface with boundary, if $E(u) < \hbar$, then

$$E(u) \leq C \cdot \mathrm{length}^2_{g_J}(u(\partial\Sigma)).$$

Here, g_J is the induced metric $\omega(\cdot, J\cdot)$ from ω and J.

For this problem, one needs to state and use the monotonicity lemma directly (no need to prove this lemma). **Any other intermediate results/claims need detailed justifications.

(2) Given two J-holomorphic curves: $u_0, u_1 : (\Sigma, j) \to (M, J)$ where Σ is connected, assume they agree to infinite order at point $z_0 \in \Sigma$, then prove $u_0 \equiv u_1$. Moreover, derive from this, prove that if u is not constant, then the set of its critical points is discrete in Σ.

(3) Fix a closed symplectic manifold (M, ω) and an ω-compatible almost complex structure J. Let $u_n : (\Sigma, j) \to (M, \omega, J)$ be a sequence of J-holomorphic curve such that there exists a uniform (independent of n) constant $C > 0$ with $E(u_n) < C$. Prove that if there exists a sequence of points $\{z_n\}_{n\in\mathbb{N}}$ such that z_n converge to z and $|du_n(z_n)| \to +\infty$, then z is a bubble point of the sequence $\{u_n\}_{n\in\mathbb{N}}$.

**For this problem, feel free to use the following two results.

Lemma 0.1. *Let (X, d) be a complete metric space, $\delta > 0$, $x \in X$, and $f : X \to [0, \infty)$ be a continuous function. Then there exists some $\xi \in X$ and $\epsilon > 0$ with the following properties.*

- (i) $\epsilon \leq \delta$;
- (ii) $d(x, \xi) < 2\delta$;
- (iii) $\epsilon f(\xi) \geq \delta f(x)$;
- (iv) $2f(\xi) \geq \sup_{B_\epsilon(\xi)} f$;

where $B_\epsilon(\xi)$ is the closed ball centered at ξ with radius ϵ.

Lemma 0.2. *Let $v_n : (B_{R_n}(0), j_{\mathrm{std}}) \to (M, \omega, J)$ be a sequence of J-holomorphic maps (into a closed symplectic manifold (M, ω)), where $R_n \to \infty$. If $E(v_n) < C$ for a uniform upper bound $C > 0$ (independent of n), then there exists a subsequence of $\{v_n\}_{n\in\mathbb{N}}$ converging to a J-holomorphic map $v_\infty : (\mathbb{C}, j_{\mathrm{std}}) \to (M, J)$ in C^∞_{loc}-sense (i.e, smoothly over any compact subset of $\mathbb{C}$).*