e.g. For $D^1$ ($\simeq$ a vector field on $M$), locally at any pt $p \in M$ there exists a $\underset{\text{1-dim emb. submfd}}{\underline{\text{curve}}}$ $\gamma : (-\varepsilon, \varepsilon) \to M$ s.t. $\dot\gamma(0) \in D(\gamma(0)) = D(p)$. and $\dot\gamma(q) \in D^1(q)$. locally

e.g. (More fundamental than e.g. above).

Given $D^k$ on $M$, satisfying $\forall p$, $\exists$ an immersed $\varphi : N^k \to M$

s.t. $\varphi_*(q)(T_qN) = D^k(p)$ when $\varphi(q) = p$ $(\ast)$

by def of immersion, we know $T_qN \simeq D^k(p)$

(This immersed submfd $(N, \varphi)$ may vary along $p \in M$).

Here is an insightful observation: for any $X, Y \in D^k$ (i.e. $X(p), Y(p) \in D^k(p)$),

then for $\underline{q \in N}$, s.t. $\varphi(q) = p$ $X(p)$ $(\in D^k(p))$ $\xrightarrow{\varphi_*^{-1}}$ $\tilde X(q) \in T_qN$. Similarly for $Y(p) \to \tilde Y(q)$

In other words, $\exists$ v.f.s $\tilde X, \tilde Y$ on $N$ s.t. $\varphi_*(\tilde X) = X$ and $\varphi_*(\tilde Y) = Y$.

Then recall $\varphi_*[\underbrace{\tilde X, \tilde Y}_{\in TN}] = [\varphi_*\tilde X, \varphi_*\tilde Y] = [X, Y] \in D^k$.

Poisson bracket on N

Poisson bracket on M

In short: If $D^k$ satisfies condition(*) above, then $\forall\ X, Y \in D^k$, $[X, Y] \in D^k$.

$\iff$ $\forall p \in M$, $\exists$ NBH $U$ of $p$ in $M$ and $k$ pointwise linearly independent v.f.s $X_1, \cdots, X_k$ on $U$ s.t. $[X_i, X_j] \in D^k|_U$.

( "$\Rightarrow$" since $TM|_U \simeq U \times \mathbb{R}^{\dim M}$ ✓

"$\Leftarrow$" $[\ ,\ ]$ is computed locally and it is bi-linear. )

e.g. In $\mathbb{R}^n$, consider distribution $D^k$ spanned by $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_k}$

Then obviously at every pt $p = (x_1, \cdots, x_n)$, consider $\mathbb{R}^k \subseteq \mathbb{R}^n$

(an embedded submfd with $\varphi$ = inclusion $(x_1, \cdots, x_k) \to (x_1, \cdots, x_k, 0 \cdots 0)$).

Then $\varphi_*(p)(T_p \mathbb{R}^k) = D^k(p)$.

The contrapositive of e.g. (above e.g.) is more useful:

$\exists\, X, Y \in D^k$ s.t. $[X, Y] \notin D^k \implies D^k$ does not satisfy $(*)$.

e.g. In $\mathbb{R}^3$, consider $D^2 = \text{span}\Big(\underbrace{\frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3}}_{X}, \underbrace{\frac{\partial}{\partial x_2}}_{Y}\Big)$, then

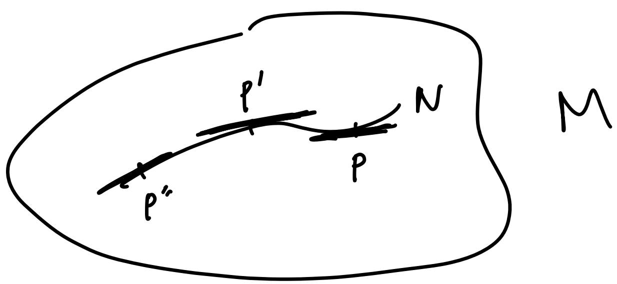
$$[X, Y] = (D_X Y^1 - D_Y X^1,\ D_X Y^2 - D_Y X^2,\ D_X Y^3 - D_Y X^3)$$

$$= (0, 0, -1) = -\frac{\partial}{\partial x_3} \notin D^2.$$

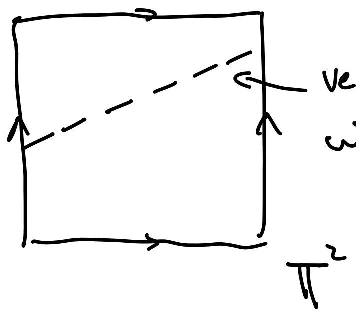
$\implies \exists$ pt $p \in M$ s.t. no immersed submfd $\varphi: N \to M$ passes through point $p$ s.t. $T\varphi(N) = D^2(\varphi)$ for those $p \in \text{im}(\varphi)$.

Question to the class: Can you figure out which pt $p$ this is?

Rmk. A more efficient way to express $T\varphi(N)$ and $D^k$ is to assume $N \subset M$ immersed submfd. Then $\forall p \in N (\subset M)$, $\text{im}(i_*(p): T_pN \to T_pM) = D^k(p)$.



e.g. Such N in Rmk above can be immersed but not embedded.




$\Longrightarrow$ $\exists$ immersed submfd but it is a dense curve on $\mathbb{T}^2$.
(so not embedded).

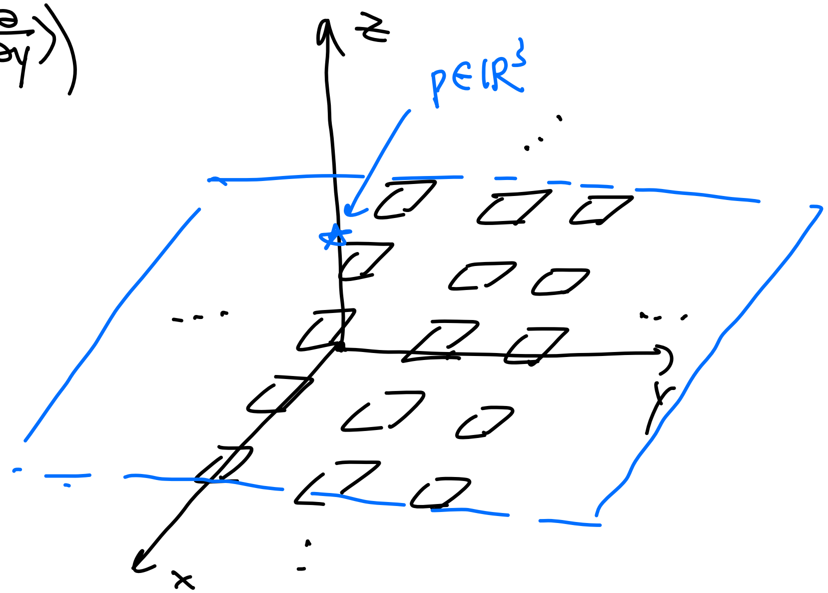
Rmk (related to e.g. above)

$$\left(\mathbb{R}^3,\ D^2 = \operatorname{span}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\right)$$

Note that

$$\mathbb{R}^3 = \text{2-plane } \langle -,-,z\rangle \times \mathbb{R}(z)$$

this structure is called a foliation of $\mathbb{R}^3$.



Thm (Frobenius integrability Thm) $(M, D^k)$.

$$\underbrace{\begin{array}{c} \forall p \in M, \exists N \subset M \text{ with } p \in N \ (\text{passing through } p) \\ \text{s.t. } T_xN \subset D^k(x) \ \forall x \in N \end{array}}_{D^k \text{ is integrable}} \iff \underbrace{\begin{array}{c} \forall X, Y \in D^k, \text{ we have} \\ [X, Y] \in D^k. \end{array}}_{D^k \text{ is involutive}}$$

$\Longleftarrow$ non-trivial part

(in short: integrable $\iff$ involutive).

Here is another way to express Thm (RHS) above, via differential forms. Hence we need to know how to transfer $D^k$ to forms.

**Def** Given $(M, D^k)$, a $p$-form $\alpha \in \Omega^p(M)$ (for $1 \le p \le \dim M$) annihilates $D^k$ if

$$\alpha(X_1, \cdots, X_p) = 0 \quad \text{for any } X_1, \cdots, X_p \in D^k.$$

$$I(D^k) = \{ \underbrace{\alpha = \alpha_1 + \cdots + \alpha_{\dim M}}_{\in \Omega^*(M)} \mid \overset{\in \Omega^p(M)}{\alpha_p} \text{ annihilates } D^k \}.$$

Observations

- $I(D^k)$ is an ideal (under the multiplication wedge $\wedge$)

$$\underset{\underset{\Omega^q(M)}{\cap}}{(\beta} \wedge \alpha_p)(X_1, \cdots, X_p, X_{p+1}, \cdots X_{p+q}) = \sum \alpha_p(\cdots) \cdot \square = 0$$

by def of $\alpha_p$.

- Def of $D^k \Rightarrow \exists\ X_{\dim M-k+1} \cdots X_{\dim M} \in D^k$ linearly independent for any $x \in M$. Then locally around NBH $U$ of each pt $x \in M$, one extends

$$\{\underbrace{X_1, \cdots, X_{\dim M-k}}_{\notin D^k}, \underbrace{X_{\dim M-k+1}, \cdots, X_{\dim M}}_{\in D^k}\}$$

linearly independent over $U$.

Then there duals (as 1-forms)

$$\{\alpha_1, \cdots, \alpha_{\dim M-k}, \alpha_{\dim M-k+1}, \cdots, \alpha_{\dim M}\}$$ form a basis of $\Omega^1(U)$

where $\{\alpha_1, \cdots, \alpha_{\dim M-k}\}$ are linearly independent and generate an ideal $I(U)$

in $\Omega^*(U)$.

$\Rightarrow$ if $\alpha \in I(D^k)$, then $\alpha|_U \in I(U)$.

$\underbrace{X_1 \quad X_2}_{\notin D^2} \quad \underbrace{X_3}_{\in D^2}$

$\alpha_{\dim M - k}$

(Indeed, for brevity, if $\{\alpha_1, \dots, \alpha_{\dim M}\} = \{\alpha_1, \underset{3-1}{\alpha_2}, \alpha_3\}$

then $\alpha|_U = f_1\alpha_1 + f_2\alpha_2 + f_3\alpha_3 + f_{12}\alpha_1\wedge\alpha_2 + f_{13}\alpha_1\wedge\alpha_3 + f_{23}\alpha_2\wedge\alpha_3 + f_{123}\alpha_1\wedge\alpha_2\wedge\alpha_3$

$(\alpha|_U)(X_3) = 0 \Rightarrow f_3 = 0$, so $\alpha|_U \in I(U)$.)

Prop $D^k$ is involutive iff $I(D^k)$ satisfies $\underbrace{\underbrace{dI(D^k)}_{=\{d\alpha \mid \alpha \in I(D^k)\}} \subset I(D^k)}_{I(D^k) \text{ is a differential ideal.}}$

$\Rightarrow$ $D^k$ is integrable $\Leftrightarrow$ $I(D^k)$ is a differential idea.

By Frobenius integrability thm

e.g. Recall $D^2 = \mathrm{span}\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \quad \frac{\partial}{\partial y}\right)$ in $\mathbb{R}^3$, then compute $I(D^2)$.

$X_2 = (1, 0, y)$, $X_3 = (0, 1, 0)$ spans $D^2$.

$\xRightarrow[\text{extend}]{\text{Locally}}$ $X_1 = (0, 0, 1)$ $\quad X_2 = (1, 0, y)$ $\quad X_3 = (0, 1, 0)$ basis

$\xRightarrow{\text{dual}}$ $\alpha_1 = dz - y\,dx$ $\quad \alpha_2 = dx$ $\quad \alpha_3 = dy$ dual basis

$\Longrightarrow$ $I(D^2)$ is the ideal generated by $\alpha_1 = dz - y\,dx$

Let's verify $I(D^2)$ is a differential ideal or not:

$$d\alpha = d(dz - y\,dx) = -dy \wedge dx = dx \wedge dy \; \left( \overset{?}{=} \alpha \wedge \square \right)$$ Impossible!

Therefore $D^2$ is NOT integrable.

b/c $d\alpha = \alpha \wedge \beta \Rightarrow d\alpha \wedge \alpha = \alpha \wedge \alpha \wedge \beta = 0$.

Rmk: If $I(D^1) = \langle \alpha \rangle$ is a differential ideal, then $d\alpha \wedge \alpha \equiv 0$.

Rmk For a 3-dim mfd $M$, any 1-form $\alpha$ s.t $d\alpha \wedge \alpha \neq 0$ for any pt on $M$, then $\alpha$ is called a contact 1-form.

so $d\alpha \wedge \alpha$ is a volume form

called completely un-integrable.

$\Longrightarrow$ $\ker\alpha$ is not integrable. (called a <u>contact structure</u>).

<u>Pf of Prop</u>: "$\Rightarrow$" For $\alpha \in I(D^k)$ (and $\alpha \in \Omega^\ell(M)$), then

$$d\alpha(X_0, X_1, \cdots, X_p) = \underbrace{\sum_i (-1)^i X_i \alpha(X_0 \cdots \hat{X}_i \cdots, X_p)}_{=0 \text{ b/c } \alpha \in I(D^k)} + \sum_{i<j} (-1)^{i+j} \alpha(\underbrace{[X_i, X_j]}_{\in D^k \text{ b/c } D^k \text{ is involutive}}, X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_p)$$

$\Longrightarrow d\alpha(X_0, \cdots, X_p) = 0$.

"$\Leftarrow$" For $X, Y \in D^k$, at every fixed pt $x \in M$, nearby we have a (dual) basis

$$\{\underbrace{\alpha_1, \cdots, \alpha_{\dim M - k}}_{\text{locally span } I(D^k)}, \alpha_{\dim M - k+1}, \cdots, \alpha_{\dim M}\}$$

Extend $\alpha_i$'s by zero, so these 1-forms are defined over $M$. Then

$$\alpha_i([X, Y]) = X\alpha_i(Y) - Y\alpha_i(X) - \underbrace{d\alpha_i}_{\in I(D^k) \text{ by assumption}}(X, Y) \Rightarrow \alpha_i([X, Y]) = 0$$

Since this holds for any $\alpha_i$ for $1 \leq i \leq \dim M - k$, we know

$[X, Y] \in D^k$ $\square$

For the proof of Frobenius integrability thm, see Lundell's paper:

A short proof of the Frobenius thm 1992 (PAMS). 2.5 pages.

## 3. Sard's Thm

Recall $F: N \to M$ smooth map has its derivative at $p$ classified as the following two case:

full rank is an open condition (cf. constant rank thm)

① $dF(p): T_pN \to T_{F(p)}M$ rank $dF(p) = \dim M$.

then $p$ is called a regular pt and $F(p) \in M$ is called a <u>regular value</u>

② $dF(p): T_pN \to T_{F(p)}M$ rank $dF(p) < \dim M$

then $p$ is called a critical pt and $F(p) \in M$ is called a <u>critical value</u>

Rmk Any $x \notin \operatorname{Im} F$ is also called a regular value.

A. Morse

Thm (Morse-Sard) For smooth $F: N \to M$, the set of critical values has (Lebegue) measure $0$.

locally, $\operatorname{im}(F)$ can be covered by balls with arbitrarily small total volume in $\mathbb{R}^{\dim M}$.

Rmk: If $F$ is not smooth, say only continuous, then $\exists$ space-filling curve $p: \mathbb{R} \to [0,1]^2$ that has positive measure! (see Peano's curve).

Rmk: Consider constant map $F: N \to \{pt\} \in M$. Then the set of critical points could have large measure.

Rmk: For smooth $F: N \to M$, the set of critical value could be dense (but measure $0$, e.g. $\mathbb{Q} \subset \mathbb{R}$).

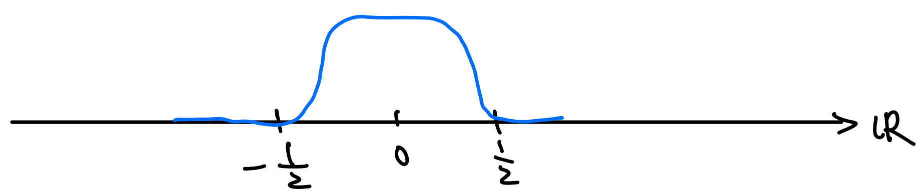
eg. list rational number by $r_1, r_2, \cdots$.

Then fix $f: \mathbb{R} \to \mathbb{R}$ in the shape of

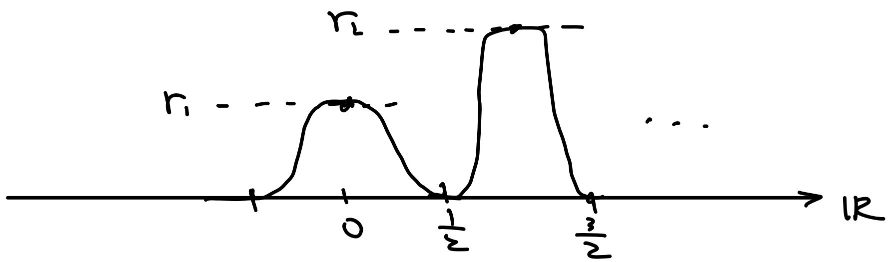
$f(0) = 1$

$f'(0) = 0$.

$f$ cc $(-\frac{1}{2}, \frac{1}{2})$.




Then consider $F: \mathbb{R} \to \mathbb{R}$ by $F(x) = \sum_{i \in \mathbb{N}} r_i f(x-i)$ $\Leftarrow$ This fcn makes sense.




Then $F'(x) = 0 \Longrightarrow$ many points but $F(\text{critical pts}) = \{r_1, r_2, \cdots\}$ dense in $\mathbb{R}$.

__Rmk.__ Sard's thm can be generalized to $\infty$-ly dim'l case.

Recall a set $A$ in a topological space is called __residual__ if it is an intersection of countably many dense open subset.

e.g. $\mathbb{R} \setminus \mathbb{Q}$ is residual but $\mathbb{Q}$ is <u>not</u> residual.

<u>Thm (Smale, 1965)</u> If $\underbrace{F: M}_{\text{could be only dim'l}} \to N$ smooth map, then the set of
(smooth version)

regular values is a residual set in $N$.

In math, one say "the property holds <u>generically</u>" meaning that the objects that satisfy this property form a residual set.

$\Rightarrow$ $F: M \to N$, then point in $N$ is a regular value generically.

- A direct cor of Sard's thm: if $F: N \to M$ and $\dim N < \dim M$, then measure of $F(N)$ is zero in $M$.

<u>pf</u>. If measure of $F(N)$ is not zero, then it must contain at least one regular value $r \in F(N)$, By pre-image prop (coming from constant rank thm), the pre-image $F^{-1}(r)$ is a submfd of $\dim N - \dim M$.

$\to \leftarrow$.

* Therefore, we have seen that one application of Sard's thm is to guarantee the existence of at least one regular value.

- Here is another application.

$$F: N \longrightarrow M \qquad \text{smooth and } \underline{\text{proper}}$$

$N$: orientable connect without bd

and $\dim N = \dim M \; (= n > 0)$

Recall degree of $F$ is defined by fixing any $\alpha \in H^n_c(M;\mathbb{R})$ and

$$\deg(F) = \int_N F^*\alpha \Big/ \int_M \alpha$$

well-defined b/c $F$ is proper

Recall if $F$ is <u>not</u> surjective, then $\deg(F) = 0$.

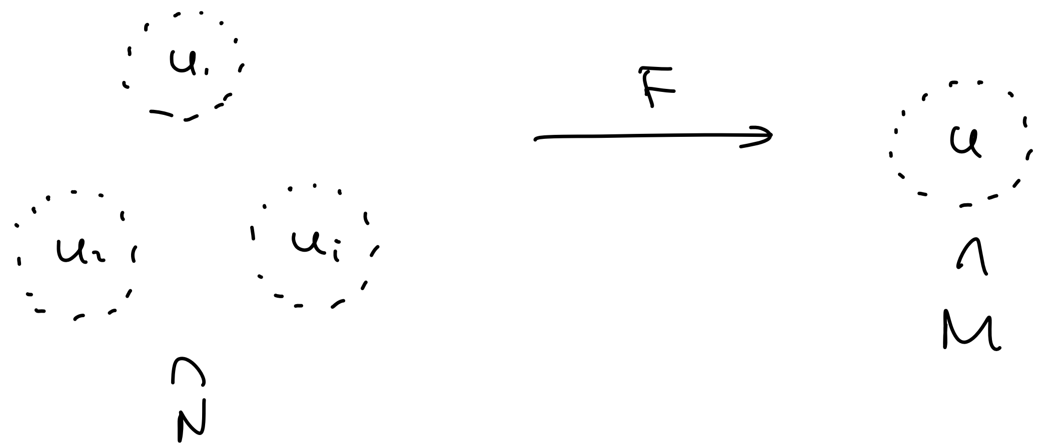
Assume $F$ is surjective, and then Sard's Thm $\Rightarrow$ $\exists$ at least one regular value $p \in \mathrm{Im}(F)$.

$\Rightarrow$ $\quad F^{-1}(\{p\})$ is cpt submfd of degree $\dim N - \dim M = 0$

$\Rightarrow \quad F^{-1}(\{p\}) = \{q_1, \dots q_n\}$ finitely many pts.

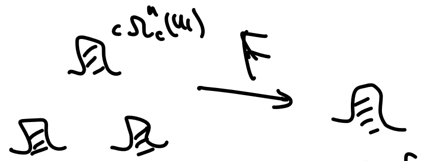
Moreover, choose a sufficiently small NBH $U$ of $p$,

$$F^{-1}(U) = U_1 \sqcup \cdots \sqcup U_n \quad \text{where } U_i \text{ is a NBH of } q_i$$



and $F|_{U_i} : U_i \overset{\text{diffeo}}{\simeq} U \Rightarrow$ each $i$ associates $\sigma_i = \pm 1$ depending whether $F$ preserve orientation or not.

Then claim: $\deg(F) = \sum_{i=1}^{n} \sigma_i$

b/c 

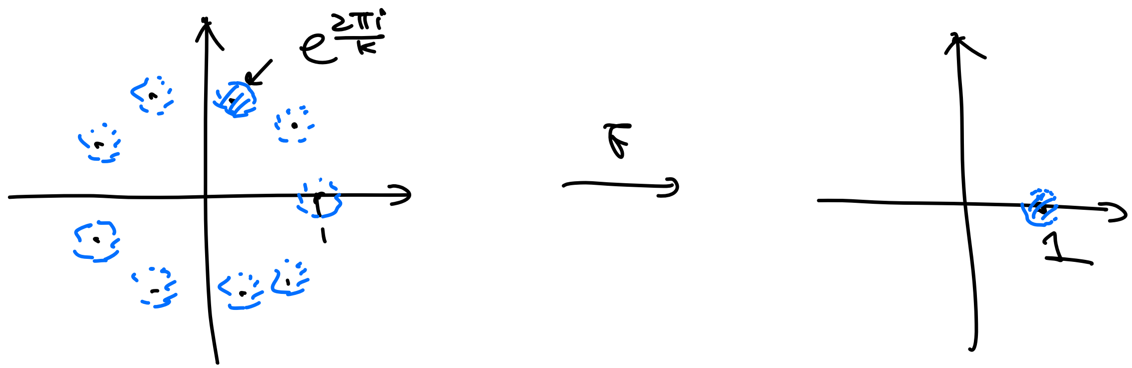

$\alpha = [\theta]$

$\theta \in \Omega^n_c(U)$ and $\int_M \alpha = 1$

Then $\int_N F^*\alpha = \sum_{i=1}^n \int_{U_i} F^*\alpha$

$= \sum_{i=1}^n \sigma_i \int_U \alpha = \sum_{i=1}^n \sigma_i$ $\square$

e.g. $F : \mathbb{C} \to \mathbb{C}$ $\quad z \to z^k$. Consider any regular value, say $1$.




$$\deg(F) = \underbrace{1 + 1 + \cdots + 1}_{k} = k.$$

- Here is the third application.

For $F : N \to M$, if $q \in N$ is a critical pt (i.e. $dF(q)$ is not full rank), then one usually don't know how "bad" it could be.

e.g. $F: \mathbb{R}^2 \to \mathbb{R}$

- $F(x,y) = x^2+y^2 \rightsquigarrow \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right) = (2x, 2y)$

  $\Longrightarrow$ crit point is only $(0,0)$.

- $F(x,y) = x^2y^2 \rightsquigarrow \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right) = (2xy^2, 2x^2y)$

  $\Longrightarrow$ $(0,0)$ is a critical pt (and there are more).

These two cases are fundamentally different.

Consider $\tau(f): \mathbb{R}^2 \to \mathbb{R}^2$ by $(x,y) \mapsto \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$

Then $(x,y)$ is a critical point of $f$ iff $\tau(f)(x,y) = (0,0)$

and $d\tau(f)$ is just the Hessian of $f$

- $d^2(f)(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ non-deg matrix
- $d^2(f)(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ deg matrix.

$\Rightarrow$ In the first case, $\nabla(f)$ is locally a diffeo of NBH of $(0,0)$ to the $(0,0) \in \mathbb{R}^2$.

critical pt

$\Leftarrow$ around $(0,0)$, the pt $(0,0)$ is the only critical point of function $f: \mathbb{R}^2 \to \mathbb{R}$.

In the second case, it is not.

<u>Def</u>. Let $F: M \to \mathbb{R}$ be a smooth fcn. A critical pt $p \in M$ is <u>non-deg</u> if locally $\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)(p)$ is a non-deg matrix.

If all critical pts of $F$ are non-deg, then $F$ is called a Morse fcn

M. Morse

<u>Prop</u> If $F: U(\subset \mathbb{R}^n) \to \mathbb{R}$ is a smooth fcn, then for a generic $\vec{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$, the function

$$f_{\vec{a}}(x) = f(x) - a_1 x_1 - \cdots - a_n x_n$$

is Morse.

~~Pf~~ Still use $\tau(f)$: $U^{\subset\mathbb{R}^n} \to \mathbb{R}^n$

$$x=(x_1,\dots,x_n) \longmapsto \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right)$$

Then $df_{\vec{a}}(x) = \left(\frac{\partial f_{\vec{a}}}{\partial x_1}(x), \dots, \frac{\partial f_{\vec{a}}}{\partial x_n}(x)\right)$

$$= \left(\frac{\partial f}{\partial x_1}(x)+a_1, \dots, \frac{\partial f}{\partial x_n}(x)+a_n\right) = \tau(f)(x) - \vec{a}.$$

Then $x$ is a critical pt of $f_{\vec{a}}$ iff $\tau(f)(x) = \vec{a}$. generically exists.

For (smooth) map $\tau(f)$, take a regular value $\vec{a} \in \mathbb{R}^n$, then by def, $d\tau(f)(x) = \mathrm{Hess}(f)(x) = \mathrm{Hess}(f_{\vec{a}})(x)$ is non-deg, for any $x$ s.t. $\tau(f)(x) = \vec{a}$.

$\Rightarrow$ $x$ —
- $\tau(f)(x) = \vec{a} \Rightarrow x$ is a crit pt of $f_{\vec{a}}$
- $\mathrm{Hess}(f_{\vec{a}})(x)$ non-deg $\Rightarrow x$ is a non-deg crit pt.

$\Rightarrow f_{\vec{a}}$ is Morse.

Rmk Generically, any smooth fcn is Morse.

Ref: ① Proof of Sard's Thm, see Chapter 3 in Milnor's "Topology from the differentiable viewpoint."

② Morse fcn $\Rightarrow$ homological theory (Morse theory)

see "Morse theory" by Milnor.

---

End / 12/31/2024