Lecture 8 Miscellaneous ← last lecture of this semester

This lecture covers 3 different topics

1. Hodge theory (without analysis part)
2. integrable distribution
3. Sard's thm

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1. Hodge theory (de Rham-oriented).

Recall/Define the $*$-operator: fix a volume form Ω of an oriented mfd M^n. Locally suppose space of forms has the following basis

$e^1 = dx_1$ in coordinate

$$\{e^1, \dots, e^n, \; e^{i_1} \wedge e^{i_2}, \dots, \; e^{i_1} \wedge \cdots \wedge e^{i_k}, \dots e^1 \wedge \cdots \wedge e^n\}.$$

Then

$$*(e^{i_1} \wedge \cdots \wedge e^{i_k}) = (-1)^{\Sigma i_j - i_k} e^1 \wedge \cdots \wedge \widehat{e^{i_1}} \wedge \cdots \wedge \widehat{e^{i_k}} \wedge \cdots \wedge e^n$$

where $\varepsilon_{i_1 \cdots i_k} = (-1)^{i_1 + \cdots + i_k + 1 + \cdots + k}$.

(or equivalently, $\varepsilon_{i_1 \cdots i_k}$ is determined by

positive.

$$e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge *(e^{i_1} \wedge \cdots \wedge e^{i_k}) = +\Omega, \quad)$$

assuming locally $\Omega = e^1 \wedge \cdots \wedge e^n$

In general, if $\underset{\Omega^k(M)}{\alpha} = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}$, then define

$$*\alpha = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k} * (e^{i_1} \wedge \cdots \wedge e^{i_k}).$$

This makes $*$-operator to be a well-defined operation on tensors.

In this way, $*: \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$ a $C^\infty(M)$-linear map.

Observations.

- For $\alpha \in \Omega^k(M)$, then $**\alpha = (-1)^{k(n-k)} \alpha$

$$*\left(*(e^{i_1} \wedge \cdots \wedge e^{i_k})\right) = (-1)^{i_1 + \cdots + i_k + 1 + \cdots + k} * (e^1 \wedge \cdots \wedge \widehat{e^{i_1}} \wedge \cdots \wedge \widehat{e^{i_k}} \wedge \cdots \wedge e^n)$$

$$= (-1)^{(1+\cdots+n)+(1+\cdots+k+1+\cdots+n-k)} e^{i_1}\wedge\cdots\wedge e^{i_k}$$

$$= (-1)^{\frac{(1+n)n}{2}+\frac{(1+k)k}{2}+\frac{(1+n-k)(n-k)}{2}} e^{i_1}\wedge\cdots\wedge e^{i_k}$$

$$= (-1)^{\frac{n^2+n+k^2+k+(n-k)+n^2-2nk+k^2}{2}} e^{i_1}\wedge\cdots\wedge e^{i_k}$$

$$= (-1)^{\overbrace{n^2+n}^{\text{always even}}+k^2-nk} e^{i_1}\wedge\cdots\wedge e^{i_k}$$

$$= (-1)^{k(n-k)} e^{i_1}\wedge\cdots\wedge e^{i_k}.$$

- $\underset{\substack{\uparrow \\ \text{constant function}}}{* 1} = e^1\wedge\cdots\wedge e^n = \Omega.$ $\quad *\Omega = *(e^1\wedge\cdots\wedge e^n) = (-1)^{1+\cdots+n+1+\cdots+n}\, 1 = 1.$

One can introduce a <u>local</u> inner product w.r.t basis above.

$$\underset{\substack{\uparrow \\ \Omega^k(M)}}{\langle \alpha, \beta \rangle} := \langle \sum f_{i_1\cdots i_k} e^{i_1}\wedge\cdots\wedge e^{i_k}, \sum g_{j_1\cdots j_k} e^{j_1}\wedge\cdots\wedge e^{j_k} \rangle$$

$$= \sum_{i_1<\cdots<i_k} f_{i_1\cdots i_k}\, g_{i_1\cdots i_k}$$

$\leftarrow$ this is commutative $\langle \alpha, \beta\rangle = \langle \beta, \alpha \rangle$

Prop. $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \Omega$ for $\alpha, \beta \in \Omega^k(M)$

Pf. $\alpha = \sum f_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}$

$$*\beta = \sum \pm g_{j_1 \cdots j_k} \underbrace{e^1 \wedge \cdots \wedge \widehat{e^{j_1}} \wedge \cdots \wedge \widehat{e^{j_k}} \wedge \cdots \wedge e^n}_{= *(e^{j_1} \wedge \cdots \wedge e^{j_k})}$$

$$\Rightarrow \alpha \wedge *\beta = \sum \pm f_{i_1 \cdots i_k} g_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge *(e^{i_1} \wedge \cdots \wedge e^{i_k})$$

$$= \langle \alpha, \beta \rangle \Omega.$$

therefore

$$\begin{aligned} \langle *\alpha, *\beta \rangle \Omega &= *\alpha \wedge (**\beta) \\ &= *\alpha \wedge (-1)^{k(n-k)} \beta \\ &= (-1)^{k(n-k)+k(n-k)} \beta \wedge (*\alpha) \\ &= \langle \beta, \alpha \rangle \Omega \\ &= \langle \alpha, \beta \rangle \Omega \end{aligned}$$

$$\Rightarrow \langle *\alpha, *\beta \rangle = \langle \alpha, \beta \rangle.$$

By discussion above, the interesting object is $\alpha \wedge *\beta$. Now, take $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$, then

$$\underbrace{d(\alpha \wedge *\beta)}_{\in \Omega^n(M)} = d\alpha \wedge (*\beta) + (-1)^{k-1} \alpha \wedge \underbrace{d*\beta}_{\in \Omega^{n-k+1}}.$$

$$= d\alpha \wedge (*\beta) + (-1)^{k-1+(n-k+1)(k-1)} \alpha \wedge (*(*d*\beta))$$

$$= d\alpha \wedge (*\beta) + (-1)^{n(k-1)} \alpha \wedge (*(*d*\beta))$$

$$= d\alpha \wedge (*\beta) - \alpha \wedge * (((-1)^{n(k-1)+1} *d*)\beta)$$

Denote $\underline{\underline{\delta := (-1)^{n(k-1)+1} *d*}}$. Then for a closed mfd M, we have

$$\underset{\substack{\uparrow \\ \text{Stokes' Thm}}}{0} = \int_M d\alpha \wedge (*\beta) - \int_M \alpha \wedge (*\delta\beta)$$

$$\iff \quad \int_M \langle d\alpha, \beta \rangle \Omega = \int_M \langle \alpha, \delta\beta \rangle \Omega \qquad (*)$$

Define a global inner product on space of forms (w.r.t a fixed volume form Ω) by

locally defined via fcns.

$$\underset{\in \Omega^k(M)}{(\alpha, \beta)} := \int_M \langle \alpha, \beta \rangle \, \Omega$$

If $\deg\alpha \neq \deg\beta$, then $\langle \alpha, \beta \rangle = 0$.

DIY, one can check that this is ind of the choice of local coordinate.

Then (*) above can be neatly reformulated by, $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^k(M)$,

$$(d\alpha, \beta) = (\alpha, \delta\beta)$$

(and $(\delta\alpha, \beta) = (\alpha, d\beta)$). for $\alpha \in \Omega^{k+1}(M)$ and $\beta \in \Omega^k(M)$

This means, w.r.t the inner product $(\,,\,)$, δ is the adjoint operator of d.

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M) \quad \text{but} \quad \delta : \Omega^k(M) \longrightarrow \Omega^{k-1}(M).$$

Def Hodge-Laplace operator is defined by

$$\Delta := \delta d + d\delta$$

Note $\Delta : \Omega^k(M) \to \Omega^k(M)$.

Observations:

- Δ is self-adjoint, i.e. $(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$.

$$(\Delta\alpha, \beta) = (\delta d\alpha + d\delta\alpha, \beta) = (d\alpha, d\beta) + (\delta\alpha, \delta\beta) = (\alpha, \Delta\beta).$$

$\Rightarrow$ If $\alpha = \beta$, then

$$(\Delta\alpha, \beta) = (\Delta\alpha, \alpha) = (d\alpha, d\alpha) + (\delta\alpha, \delta\alpha) \geq 0.$$

Moreover, if $(\Delta\alpha, \alpha) = 0$, then both $d\alpha = 0$ and $\delta\alpha = 0$, so $\Delta\alpha = 0$.

$\xRightarrow{\text{conclusion}}$ $\Delta\alpha = 0$ iff $d\alpha = \delta\alpha = 0$.

Def For a mfd M, its harmonic k-form is

$$\mathcal{H}^k(M) := \{ \alpha \in \Omega^k(M) \mid \Delta\alpha = 0 \}$$

$$\left(= \{ \alpha \in \Omega^k(M) \mid d\alpha = \delta\alpha = 0 \} \right).$$

Note that $\mathcal{H}^k(M) \subset$ closed k-form (so it reps some de Rham coh class).

Geometric meanings of a harmonic forms: for $\alpha \in \mathcal{H}^k(M)$, for any $\sigma \in \Omega^{k-1}(M)$.

$$(\alpha + d\sigma, \alpha + d\sigma) = (\alpha,\alpha) + 2(\alpha, d\sigma) + (d\sigma, d\sigma)$$

a cute step $\longrightarrow$ $$= (\alpha,\alpha) + \cancel{2(\delta\alpha, \sigma)}^{\,0} + (d\sigma, d\sigma)$$

$$= (\alpha,\alpha) + \underbrace{(d\sigma, d\sigma)}_{\geq 0}$$

$$\geq (\alpha,\alpha).$$

$\Rightarrow$ Within all possible reps of deRham coh class θ, the harmonic rep is the "minimal" one w.r.t inner product $(\cdot,\cdot)$.

<u>Thm (Hodge)</u> Let M be a closed mfd.

- $\mathcal{H}^k(M)$ is finite dim'l over $\mathbb{R}$.
- Any $\alpha \in \Omega^k(M)$ can be uniquely decomposed into

$$\alpha = \underset{\mathcal{H}^k(M)}{\underset{\cap}{\alpha_h}} + \underbrace{d\sigma}_{\in\,\Omega^{k-1}(M)} + \delta\sigma' \rightarrow \Omega^{k+1}(M).$$

$\leftarrow$ Hodge decomposition

If α is closed, then
$0 = d\alpha = d\delta\sigma'$
$\Rightarrow (\delta\sigma', \delta\sigma') = (\sigma', d\delta\sigma') = 0$
$\Rightarrow \delta\sigma' = 0$
$\Rightarrow \alpha = \alpha_h + d\sigma$

How to apply such a deep thm:

<u>e.g.</u> consider (M is closed)

$$i: \mathcal{H}^k(M) \longrightarrow H^k_{dR}(M)$$
$$\alpha \longrightarrow [\alpha]$$

Then if $\overset{i(\alpha)}{[\alpha]} = \overset{i(\beta)}{[\beta]}$, then $\beta = \alpha + d\sigma$. then $(\beta,\beta) \geq (\alpha,\alpha)$.

(and symmetrically, $(\alpha,\alpha) \geq (\beta,\beta)$), so $\alpha = \beta$. This means i is injective.

For any class $[\alpha] \in H^k_{dR}(M)$, by Hodge decomposition,

$\alpha = \alpha_h + d\sigma + \delta\sigma'$ (where $\delta\sigma' = 0$ b/c α is closed)

Then $[\alpha_h] = [\alpha]$. This implies i is also surjective.

$\Longrightarrow$ $H^k_{dR}(M; \mathbb{R})$ is finite dim'l over $\mathbb{R}$.
by Hodge thm

In particular every deRham coh class always admits a harmonic rep.

e.g. Claim: $*\Delta = \Delta *$

Pf: suffice to show $*d\delta = \delta d *$ (the by a symmetric argument, $*\delta d = d\delta *$. $\Rightarrow *\Delta = *(d\delta + \delta d) = \cdots = \Delta *$).

For any $\alpha \in \Omega^k(M)$,

$$\begin{aligned} *\delta\alpha &= *(-1)^{n(k-1)+1} * d * \alpha \\ &= (-1)^{n(k-1)+1} (-1)^{(n-(n-k+1))(n-k+1)} d * \alpha \\ &= (-1)^{-k^2} d * \alpha = (-1)^k d * \alpha \end{aligned}$$

Similarly,

$$\underbrace{\delta *\alpha}_{\in \Omega^{n-k}(M)} = (-1)^{n(n-k-1)+1} * d * (*\alpha)$$

$$= (-1)^{n(n-k-1)+1+k(n-k)} * d\alpha = (-1)^{k+1} * d\alpha$$

$$\Rightarrow \quad * d \underbrace{\delta\alpha} = (-1)^{k} \delta * \delta\alpha = (-1)^{k+k} \delta (d * \alpha) = \delta d * \alpha \qquad \square$$

This relation $*\Delta = \Delta *$ implies that if $\alpha \in \mathcal{H}^k(M)$, then

$*\alpha \in \mathcal{H}^{n-k}(M)$.

Then by the previous e.g. we have for closed mfd M,

$$H^k_{dR}(M;\mathbb{R}) \simeq \mathcal{H}^k(M) \xrightarrow[\simeq]{*} \mathcal{H}^{n-k}(M) \simeq H^{n-k}_{dR}(M;\mathbb{R})$$

and $*$ is an iso (b/c $** = \pm 1$). This implies the Poincaré-duality for closed mfd M.

Rmk. For the proof of Hodge thm, see chapter 6 of Warner's book or Usher's notes Chapter 1 The Hodge thm and Sobolev spaces.

2. Integrable distribution

Recall a vector field $X \in \Gamma(M)$ can be identified with a 1-dim'l subspace $\underbrace{\mathbb{R}\langle X(p)\rangle}_{\text{line spanned by } X(p)}$ in each T_pM.

An obvious generalization: take $\underset{1 \le k \le \dim M}{k}$-dim'l subspace $V_p \subset T_pM$.

Then the collection of $\{V_p\}_{p\in M}$ is called a k-dim'l distribution on M, usually denoted by D (or D^k) and $D(p) = V_p$.

(Other names: field of k-dim'l subspaces, k-dim'l subspace field...)

Here are two examples that shows D's can be different.

e.g. For D^1 ($\simeq$ a vector field on M), locally at any pt $p \in M$ there exists

a $\underset{\text{1-dim emb. submfd}}{\underline{\text{curve}}}$ $\gamma : (-\varepsilon, \varepsilon) \to M$ s.t. $\dot\gamma(0) \in D(\gamma(0)) = D(p)$. and $\underset{\text{locally}}{\dot\gamma(q) \in D^1(q)}$.

e.g. (More fundamental than e.g. above).

Given D^k on M, satisfying $\forall p$, $\exists$ an immersed $\varphi : N^k \to M$

s.t. $\varphi_*(q)(T_qN) = D^k(p)$ when $\varphi(q) = p$ (*)

by def of immersion, we know $T_qN \simeq D^k(p)$

(This immersed subbundle (N, φ) may vary along $p \in M$).

Here is an insightful observation: for any $X, Y \in D^k$ (i.e. $X(p), Y(p) \in D^k(p)$),

then for $\underline{q \in N}$, s.t. $\varphi(q) = p$ $X(p)\ (\in D^k(p)) \xrightarrow{\varphi_*^{-1}} \tilde X(q) \in T_qN$. Similarly for $Y(p) \to \tilde Y(q)$

In other words, $\exists$ v.fs $\tilde X, \tilde Y$ on N s.t. $\varphi_*(\tilde X) = X$ and $\varphi_*(\tilde Y) = Y$.

Then recall $\varphi_*[\underbrace{\tilde X, \tilde Y}_{\in TN}] = [\varphi_* \tilde X, \varphi_* \tilde Y] = [X, Y] \in D^k$.

Poisson bracket on N Poisson bracket on M