MATH5003P "Introduction to differential manifold" $=$ C^0, C^1, $\vdots$, C^∞ = smooth (main focus in this course)

main object in this course

- smooth: free to take derivatives.
- manifold vs. manifold with boundary
  manifold with corner
  stratification
  orbifold, polyfold ...
  varieties ...

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ADD 1 hr?

Course time: Monday 7:30pm-9:00pm (no break)

Tuesday 2:00pm - 3:30 pm (no break)

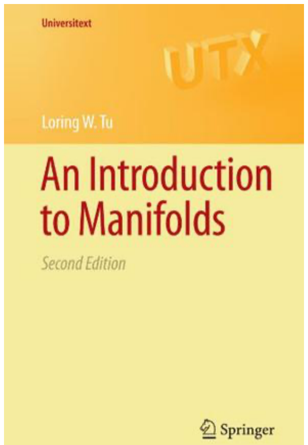
Location: 5教 102室

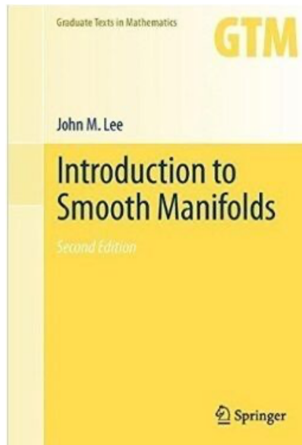
Lecturing: 中文授课 + slides in English

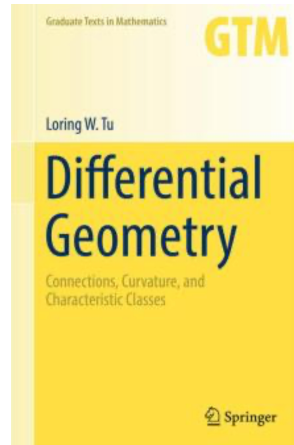
Homework: ≈Every 1.5 weeks (≈10 problems each time)

Score: 30% HW + 30% Midterm (HARD) + 40% Final

Reference

- 





- In 2023, Prof. Zuoqin Wang's (王作勤) course:

  http://staff.ustc.edu.cn/~wangzuoq/Courses/23F-Manifolds/index.html

- Remark:

  Do <u>NOT</u> copy slides (will be shared) by hand (too slow) in class!

Topics covered in this course (ideally):

	概念 concepts	计算 computations	理论 theories
EASY	- manifold and submfd - structure results of mfd	differential forms	de Rham cohomology theory
MIDDLE	- relations between mfds: vector bundles (fiber)	integration and global results	Hodge theory (on harmonic forms)
HARD	- dynamics: Lie derivatives connection	constructing examples	characteristic class theory

Extension

	概念 concepts	计算 computations	理论 theories	
EASY	- manifold and submfd - structure results of mfd	differential forms	de Rham cohomology theory	⇝ algebraic geometry
MIDDLE	- relations between mfds: vector bundles (fiber)	integration and global results	Hodge theory (on harmonic forms)	⇝ PDE & complex geometry
HARD	- dynamics: Lie derivatives connection	constructing examples	characteristic class theory	⇝ algebraic topology
	↑ memorize	↑ practice	↑ applications	

$\Rightarrow$ differential manifold is the foundation of many further developments in math.

- platform
- tools
- theorems

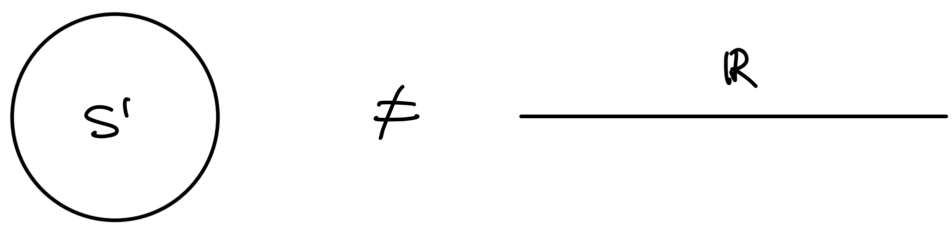
Style of lecturing

Undergraduate: concrete examples $\longrightarrow$ abstract definition

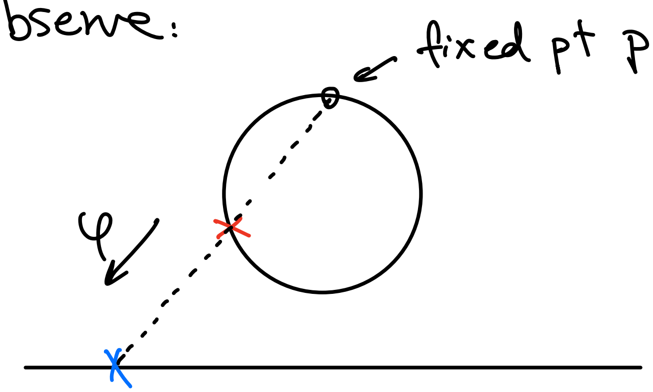
✓ Graduate : abstract definition $\longrightarrow$ concrete examples

We will take this style.

Example (Undergraduate approach)

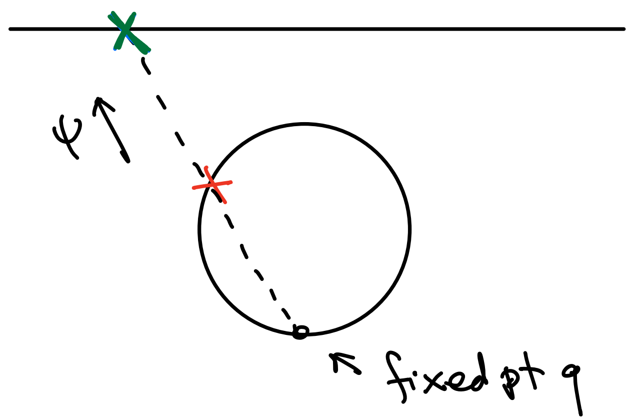


Observe:



$$S^1 \setminus \{p\} \xrightarrow[\sim]{\varphi} \mathbb{R}$$

$\Longrightarrow$ every point x $(\neq p)$ has an open neighborhood $(= S^1 \setminus \{p\})$ that looks like an open subset of $\mathbb{R}$.

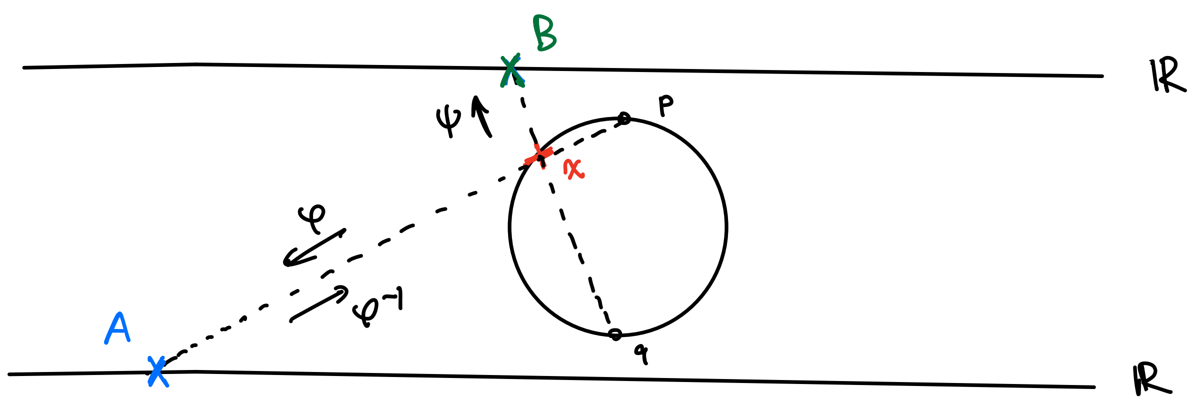


$$S^1 \setminus \{q\} \xrightarrow[\sim]{\psi} \mathbb{R}$$

$\Rightarrow$ every point $x\,(\neq q)$ has an open neighborhood $(= S^1\setminus\{q\})$ that looks like an open subset of $\mathbb{R}$.

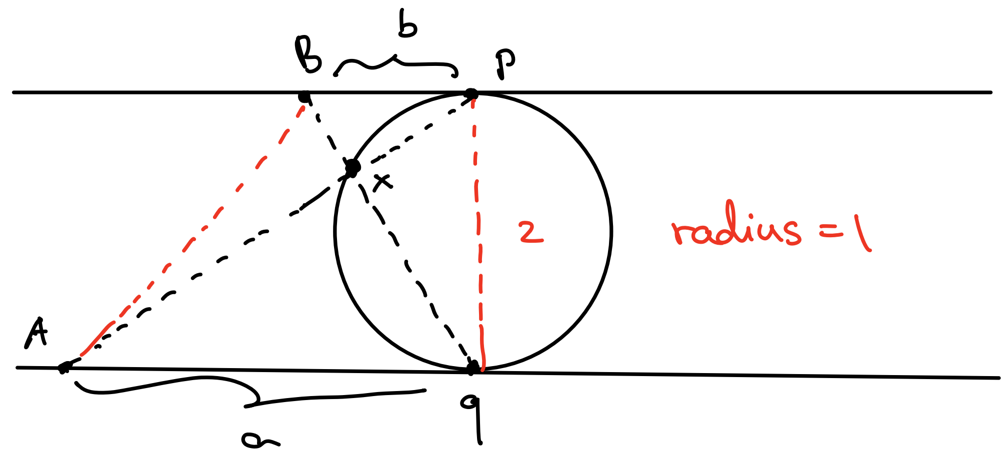
☺ Locally S^1 looks like an open subset of $\mathbb{R}$

☹ Ambiguity: For $x \neq p, q$, should I use φ or ψ to identify it to a pt in $\underline{\underline{\mathbb{R}\setminus\{0\}}}$?



$$\begin{array}{c} B \in \mathbb{R} \\ \psi \uparrow \\ x \in S^1 \\ \varphi \downarrow \uparrow \varphi^{-1} \\ A \in \mathbb{R} \end{array}$$

What is the relation between A and B ?



radius = 1

$$ab = \overline{px}\cdot\overline{xA} + \overline{Bx}\cdot\overline{xq}$$ ← elementary fact

$$= \frac{2b}{\sqrt{b^2+4}}\cdot\frac{2a}{\sqrt{b^2+4}} + \frac{2b}{\sqrt{a^2+4}}\cdot\frac{2a}{\sqrt{a^2+4}}$$

$$= 4ab\left(\frac{1}{b^2+4}+\frac{1}{a^2+4}\right)$$

coordinates in $\mathbb{R}$

$$\Longrightarrow \quad \frac{1}{a^2+4}+\frac{1}{b^2+4} = \frac{1}{4} \Longrightarrow (ab)^2 = 16 \Longrightarrow \boxed{A\cdot B = 4}$$

Therefore, $\varphi\circ\varphi^{-1}$ is the map on $\mathbb{R}\setminus\{0\}$ by "$\frac{4}{*}$".

$\Longrightarrow$ Def (of a manifold) ...

Example (Graduate approach)

Def A smooth mfd M is a second-countable Hausdorff space s.t. $\exists$ open cover $\{U_\alpha\}_{\alpha \in I}$ of M and a family of (homeomorphisms) maps $\varphi_\alpha : U_\alpha \xrightarrow{\sim} V_\alpha \subset \mathbb{K}^n$ (for some n) satisfying

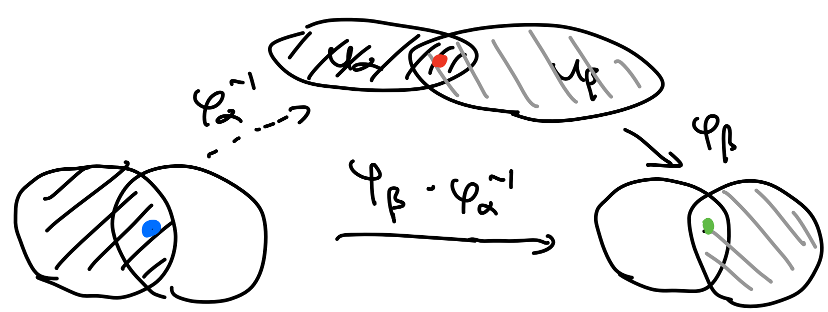
$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

for any $\alpha, \beta \in I$, the restriction

$$\varphi_\beta \circ \varphi_\alpha^{-1} : V_\alpha \cap V_\beta \longrightarrow V_\alpha \cap V_\beta$$ (transition map)

is a smooth diffeomorphism (between open subsets in $\mathbb{K}^n$)

This "n" is called the dimension of M.

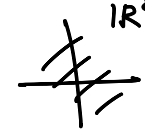


<u>Rmk</u> – If $\varphi_\beta \circ \varphi_\alpha^{-1}$ are C^r-maps, then M is called a C^r-diff mfd.

– If $\mathbb{K} = \mathbb{C}^n$ and $\varphi_\beta \circ \varphi_\alpha^{-1}$ are holomorphic maps, then M is called a complex mfd.

– Open cover $\{U_\alpha\}$ and maps $\{\varphi_\alpha\}$ are <u>not</u> unique.

<u>FACT</u>: For a smooth mfd M, $\dim(M)$ is a topological invariant under homeomorphism.

(cf. $\dim(M)$ is <u>not</u> invariant under homotopy :  $\mathbb{R}^2 \overset{htp}{\simeq} \bullet$)

e.g. (of smooth mfd)

① 0-dim'l : $M = \bigsqcup_{\text{countable}} \{p\}$ (equipped with discrete top).

$M = \left\{ \begin{matrix} & \cdot\, p_1 & \\ \cdot\, p_2 & & \cdot\, p_3 \end{matrix} \right\}$

open cover $\quad U_1 = \{p_1\} \quad U_2 = \{p_2\} \quad U_3 = \{p_3\}$

maps $\quad \varphi_1 \downarrow \quad \varphi_2 \downarrow \quad \varphi_3 \downarrow$

$\quad x \in \mathbb{R}^0 \quad x \in \mathbb{R}^0 \quad x \in \mathbb{R}^0$

No overlap $\quad U_\alpha \cap U_\beta = \phi$ for $\alpha \neq \beta$.

<u>Rmk</u> In most cases, we will consider "connected" smooth mfds.

② 1-dim'l: $\quad M = S^1$ or $M = \mathbb{R}$

They are still different, due to "compactness".

<u>Rmk</u> A mfd is called "closed" if it is cpt <u>without</u> boundary.

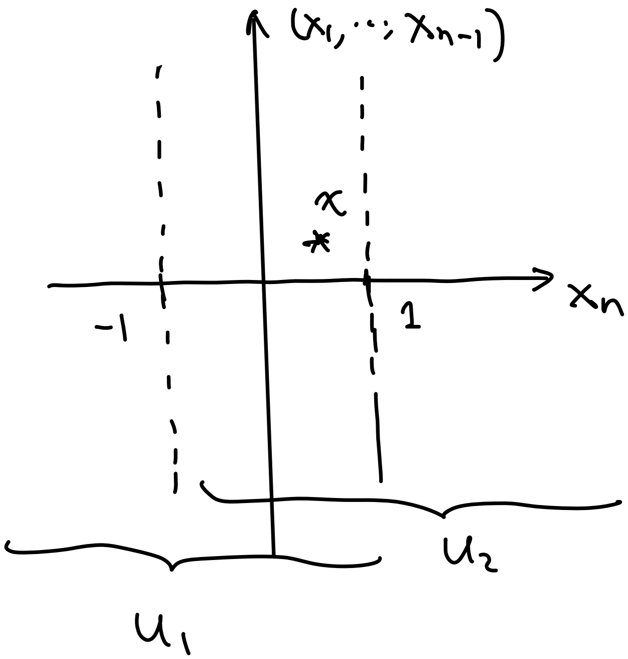
③ n-dim'l: $\quad \mathbb{R}^n$

– open cover $\quad U = \mathbb{R}^n \quad (x_1, \cdots, x_n)$

map $\quad \varphi \downarrow \qquad \downarrow$

$\quad \mathbb{R}^n \quad (x_1, \cdots, x_n)$

(standard smooth str.)



$$\mathbb{R}^n = U_1 \cup U_2$$
$$= \{x_n < 1\} \cup \{x_n > -1\}$$

$$U_1 \cap U_2 = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid -1 < x_n < 1\}.$$

$$\varphi_1 : U_1 \longrightarrow \{x_n < 1\} \subset \mathbb{R}^n$$
$$(x_1, \cdots, x_n) \rightarrow (x_1, \cdots, x_n)$$

$$\varphi_2 : U_2 \longrightarrow \{x_n > -1\}$$
$$(x_1, \cdots, x_n) \rightarrow (x_1, \cdots, -x_n)$$

Then for $x = (x_1, \cdots, x_{n-1}, x_n) \in U_1 \cap U_2$,

$$(x_1, \cdots, x_{n-1}, -x_n) \xleftarrow{\varphi_2} x \xrightarrow{\varphi_1} (x_1, \cdots, x_{n-1}, x_n)$$

$$\varphi_2 \circ \varphi_1^{-1} : (x_1, \cdots, x_{n-1}, x_n) \longrightarrow (x_1, \cdots, x_{n-1}, -x_n)$$

$\updownarrow$ matrix $\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$ (with $\underline{\det = -1}$) ← "orientation"

more e.g. ...

Example M^n smooth mfd of $\dim M = n$

$F: M \to \mathbb{R}$ a smooth fcn.

$$\begin{tikzcd} U_\alpha \arrow[r, "\varphi_\alpha"] \arrow[rr, bend right, "F|_{U_\alpha}"'] & V_\alpha \ (\subset \mathbb{R}^n) \arrow[r, "f_\alpha"] & \mathbb{R} \end{tikzcd}$$

f_α is clear.

To "glue" $\{f_\alpha\}_{\alpha \in I}$, we need compatibility: over $U_\alpha \cap U_\beta$.

$$f_\alpha \circ \varphi_\alpha \big|_{U_\alpha \cap U_\beta} = f_\beta \circ \varphi_\beta \big|_{U_\alpha \cap U_\beta}$$

Rmk. Another way is to "embed" (how?) M into $\mathbb{R}^k$ for some k.

Then define F on M by $F: \mathbb{R}^k \to \mathbb{R}$ and restricting to $M \subset \mathbb{R}^k$.

Prop: If $z \in \mathbb{R}$ is a <u>regular value</u> of F on M^n, then preimage

$$F^{-1}(z) = \{x \in M \mid F(x) = z\}$$

is a manifold of $\dim = n-1$.

This helps us to construct mfds from functions.

<u>Rmk</u> - One can consider $F: M^n \to N^{n'}$ where M, N are manifolds, then for regular value $z \in N$ of F, the preimage $F^{-1}(z)$ is a manifold of $\dim = n-n'$.

- To define "reguler value", one needs "derivatives" and it is a new concept on smooth manifolds

To simplify the situation: $F: \mathbb{R}^n \to \mathbb{R}$ and z is a regular value if $\forall x \in F^{-1}(z)$, the derivative $\left(\frac{\partial F}{\partial x_1}(x), \cdots, \frac{\partial F}{\partial x_n}(x)\right)$ is a <u>non-zero</u> vector.

e.g. $F: \mathbb{R}^n \to \mathbb{R}$ an $\overset{\neq 0}{m}$-homogeneous function:

$$F(tx_1, \cdots, tx_n) = t^m F(x_1, \cdots, x_n)$$

Then $1 (\in \mathbb{R})$ is a regular value of F.

To verify this, one apply Euler's formula for $x \in F^{-1}(1)$,

$$\frac{\partial F}{\partial x_1}(x) + \cdots + \frac{\partial F}{\partial x_n}(x) = m \cdot F(x) = m \neq 0.$$

By prop, $\{x \in M \mid F(x) = 1\}$ ($=:$ level-1 subset) is a manifold of $\dim = n-1$.

e.g. $M_{n\times n}(\mathbb{R}) = \{ n \text{ by } n \text{ matrix in } \mathbb{R}\} \approx \mathbb{R}^{n\times n}$

top on $M_{n\times n}(\mathbb{R})$ is induced by a top on $\mathbb{R}^{n\times n}$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \longleftrightarrow (a_{11}, \cdots, a_{nn})$$

$$F : M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R} \quad \text{by} \quad A \longmapsto \det(A)$$

Then $1 \ (\in \mathbb{R})$ is a regular value of F.

To verify this, we need to show for any $A \in F^{-1}(1)$, $\underbrace{\left(\frac{\partial F}{\partial a_{11}}, \cdots, \frac{\partial F}{\partial a_{nn}}\right)}_{\in \mathbb{R}^{n\times n}}(A) \neq 0$

Suffices to prove that $\exists\ V \in \mathbb{R}^{n\times n}$ s.t.

$$\left(\frac{\partial F}{\partial a_{11}}, \dots, \frac{\partial F}{\partial a_{nn}}\right)(A) \cdot V \neq 0.$$

directional derivative of F at A, along V.

inner prod.

Trick: take $V = A$.

Then

$$\frac{d}{dt} \det(A + tA)\Big|_{t=0} = \frac{d}{dt} \det((1+t)A)\Big|_{t=0}$$

$$= \det(A) \frac{d}{dt}(1+t)^n\Big|_{t=0}$$

$$= 1 \cdot \frac{d}{dt}(1 + nt + \cdots + t^n)\Big|_{t=0}$$

$$= n \neq 0$$

By prop, $\{A \in M_{n\times n}(\mathbb{R}) \mid \det(A) = 1\} =: SL(n, \mathbb{R})$ is a manifold of dim $= n^2 - 1$.

Summary (总结)

1. 课程信息

2. 课程内容 (拟)

3. 授课方式：抽象 → 具体 { 实例 / 应用 }

4. 答疑 {
助教：何政幸 李进钊
Office hr：周三下午 2:00pm – 3:00pm.
课程群：QQ群 576290693
单独联系：邮件.