HOMEWORK FOR LECTURE 7

This homework problem set can be accomplished with the help of references. Every problem worths 2 points and DO NOT LEAVE ANY PROBLEM BLANK! It is due to 11:59 pm on December 31 (sharp).

Exercise 1. Prove that any short exact sequence of cochain complexes (of $\mathbf{k}$-modules) $0 \to (C^\bullet, d_C^\bullet) \to (D^\bullet, d_D^\bullet) \to (E^\bullet, d_E^\bullet) \to 0$ induces a long exact sequence on cohomology groups,

$$\cdots \to H^*(C^\bullet; \mathbf{k}) \to H^*(D^\bullet; \mathbf{k}) \to H^*(E^\bullet; \mathbf{k}) \to H^{*+1}(C^\bullet; \mathbf{k}) \to \cdots .$$

Please provide all necessary details.

Exercise 2. Prove Künneth formula of de Rham cohomology groups. Explicitly, for manifolds M and N,

$$H^k_{\mathrm{dR}}(M \times N; \mathbb{R}) \simeq \bigoplus_{0 \le p,q \le k, p+q=k} H^p_{\mathrm{dR}}(M; \mathbb{R}) \otimes_{\mathbb{R}} H^q_{\mathrm{dR}}(N; \mathbb{R}).$$

for any $0 \le k \le \dim M + \dim N$.

Exercise 3. Compute de Rham cohomology groups (over $\mathbb{R}$) of the real projective space $\mathbb{R}P^n$ using Mayer–Vietoris sequence.

Exercise 4. Let M be a compact oriented manifold. Prove that if $\dim M = 4n+2$, then its Euler characteristic $\chi(M)$ is even.

Exercise 5. Complete following two questions on mapping degree.

(1) Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be the map $f(e^{\sqrt{-1}\theta_1}, ..., e^{\sqrt{-1}\theta_n}) = (e^{\sqrt{-1}k_1\theta_1}, ..., e^{\sqrt{-1}k_n\theta_n})$. Compute $\deg(f)$.

(2) Prove that there does *not* exist a map $S^2 \times S^2 \to \mathbb{C}P^2$ with odd degree.