HOMEWORK FOR LECTURE 6

This homework problem set can be accomplished with the help of references. Every problem worths 2 points and DO NOT LEAVE ANY PROBLEM BLANK! It is due to 11:59 pm on December 17 (sharp).

Exercise 1. For the following matrix groups $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{U}(n)$, and $\mathrm{Sp}(2n)$, compute/confirm their induced Lie algebras as follows.

(i) $\mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \,|\, \mathrm{trace}(A) = 0\}$.
(ii) $\mathfrak{sl}(n,\mathbb{C}) = \{A \in \mathfrak{gl}(n,\mathbb{C}) \,|\, \mathrm{trace}(A) = 0\}$.
(iii) $\mathfrak{o}(n) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \,|\, A^{\mathrm{T}} + A = 0\}$.
(iv) $\mathfrak{u}(n) = \{A \in \mathfrak{gl}(n,\mathbb{C}) \,|\, A^* + A = 0\}$.
(v) $\mathfrak{sp}(2n) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \,|\, A^{\mathrm{T}}J + JA = 0\}$.

Here, J is the following $(2n \times 2n)$-matrix

$$J = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix}.$$

Exercise 2. Given a Lie group G, prove the following equality:

$$\exp(-tX)\exp(-tY)\exp(tX)\exp(tY) = \exp\left(t^2[X,Y] + O(t^3)\right)$$

for any $X, Y \in \mathfrak{g}_G$, when parameter t is sufficiently small.

Exercise 3. Prove that the matrix exponential map on elements in $M_{n\times n}(\mathbb{R})$ satisfies

$$\det(e^A) = e^{\mathrm{trace}(A)}.$$

Here, $e^A = \mathbb{1} + A + \frac{A^2}{2} + \cdots$. Please provide all necessary details in your argument. Use this conclusion to confirm that the following matrix

$$\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

can *not* be written as e^A for any $A \in M_{n\times n}(\mathbb{R})$.

Exercise 4. Given a Riemannian metric g, recall that the associated curvature tensor (as a $(0,4)$-tensor) is defined by

$$R(X,Y,Z,W) := g(R(X,Y)Z,W)$$

for vector fields X,Y,Z,W. Prove the following equalities.

(i) $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$.

(ii) $R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z)$.

(iii) $R(X,Y,Z,W) = R(Z,W,X,Y)$.

Exercise 5. Consider the following (real) 2-dimensional Lie group

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \,\middle|\, x > 0,\ y \in \mathbb{R} \right\}.$$

Complete the following questions.

(i) Verify that its Lie algebra is

$$\mathfrak{g}_G = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \,\middle|\, a,b \in \mathbb{R} \right\}.$$

(ii) Take the following basis of $\mathfrak{g}_G$ in (i),

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Construct a left invariant metric g on G such that $\{X_1, X_2\}$ form an orthonormal basis.

(iii) Verify that the corresponding Levi-Civita connection ∇ of g in (ii) satisfies the following relations,

$$\nabla_{X_1} X_1 = \nabla_{X_1} X_2 = 0 \quad \nabla_{X_2} X_1 = -X_2 \quad \text{and} \quad \nabla_{X_2} X_2 = X_1.$$

(iv) Compute sectional curvatures of (G, g, ∇) for g and ∇ in (ii) and (iii).