HOMEWORK FOR LECTURE 5

This homework problem set can be accomplished with the help of references. Every problem worths 2 points and DO NOT LEAVE ANY PROBLEM BLANK! It is due to 11:59 pm on December 3 (sharp).

Exercise 1. Let M be a smooth manifold and $F : M \to \mathbb{R}^k$ be a *continuous* map. Prove that for any positive continous function $\epsilon : M \to \mathbb{R}$, there exists a smooth map $G : M \to \mathbb{R}^k$ such that $||G(x) - F(x)|| \leq \epsilon(x)$ for any $x \in M$.

Exercise 2. Consider $\theta \in \Omega^2(\mathbb{R}^3)$ defined by

$$\theta = x^2 dy \wedge dz + y dz \wedge dx + z dx \wedge dy.$$

Denote by $S^2 := \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$. Compute the integration $\int_{S^2} i^*\theta$ where $i : S^2 \to \mathbb{R}^3$ is the inclusion.

Exercise 3. (1) Given a manifold M and two 1-forms $\alpha, \beta \in \Omega^1(M)$, prove the following identity

$$\alpha \wedge (d\alpha)^n - \beta \wedge (d\beta)^n =$$

$$(\alpha - \beta) \wedge \sum_{j=0}^{n} \left((d\alpha)^j \wedge (d\beta)^{n-j}\right) + d\left(\alpha \wedge \beta \wedge \sum_{j=0}^{n-1} \left((d\alpha)^j \wedge (d\beta)^{n-1-j}\right)\right)$$

for any $n \in \mathbb{N}$. Here $(d\alpha)^n := d\alpha \wedge \cdots \wedge d\alpha$, wedged n times, similarly to others
(2) Deduce the following proposition from (1) in this exercise: Given a closed (i.e., compact without boundary) orientable manifold M of dimension $2n+1$ and smooth vector field $X \in \Gamma(TM)$, if two 1-forms $\alpha, \beta \in \Omega^1(M)$ satisfy $(\phi_X^t)^*\alpha = \alpha$ and $(\phi_X^t)^*\beta = \beta$ for any $t \in \mathbb{R}$ (invariant condition), moreover $\alpha(X) = \beta(X) = 1$, then

$$\int_M \alpha \wedge (d\alpha)^n = \int_M \beta \wedge (d\beta)^n.$$

(Note that the invariant condition above can also be expressed as $\mathcal{L}_X\alpha = \mathcal{L}_X\beta = 0$.)

Exercise 4. Let M be a closed manifold of dimension $2n$. (1) Let $\omega \in \Omega^2(M)$ be a 2-form, then ω is non-degenerate (in the sense that at any point $x \in M$, if $v \in T_xM$ is not zero, then there exists some $w \in T_xM$ such that $\omega_x(v, w) \neq 0$) if and only if ω^n is a volume form of M. Recall that a volume form means a $2n$-form that is nowhere vanishing. (2) From HW3, we have seen the (Poisson) bracket of two functions $H, G : M \to \mathbb{R}$ defined by

$$\{H, G\} := \omega(X_H, X_G), \quad \text{where } -dH = \omega(X_H, \cdot), \text{ similarly to } X_G.$$

Suppose further that ω is closed, then prove that

$$\int_M \{F, G\}\, \omega^n = 0.$$

(Hint: confirm the following equality: $\{F, G\}\, \omega^n = -n\, dG \wedge dF \wedge \omega^{n-1}$.)

Exercise 5. Let M^m, N^n be orientable manifolds. Let $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ be the projections. Then for forms $\alpha \in \Omega^m(M)$ and $\beta \in \Omega^n(N)$, consider their "product" defined by

$$\alpha \times \beta := \pi_M^* \alpha \wedge \pi_N^* \beta \in \Omega^{m+n}(M \times N)$$

prove from definition (of integration on manifold) that

$$\int_{M \times N} \alpha \times \beta = \left(\int_M \alpha \right) \cdot \left(\int_N \beta \right).$$