HOMEWORK FOR LECTURE 4

This homework problem set can be accomplished with the help of references. Every problem worths 1 point and DO NOT LEAVE ANY PROBLEM BLANK! It is due to 11:59 pm on November 22 (sharp).

Exercise 1. Consider unit open disk B^2 in $\mathbb{R}^2$ defined by $B^2 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$ equipped with the following Riemannian metric

$$g((x, y)) = \frac{4}{(1 - (x^2 + y^2))^2} \left(dx \otimes dx + dy \otimes dy \right).$$

Meanwhile, consider the open upper half plane $\mathbb{H}^2$ of $\mathbb{R}^2$, that is, $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \,|\, y > 0\}$ equipped with the following Riemannian metric

$$g'((x, y)) = \frac{1}{y^2} \left(dx \otimes dx + dy \otimes dy \right).$$

Prove that there exists a smooth diffeomorphism $F : B^2 \to \mathbb{H}^2$ such that it preserves the metrics in the sense that for any vector fields $X, Y \in \Gamma(TB^2)$, we have $g'(F_*(X), F_*(Y)) = g(X, Y)$.

Exercise 2 (GTM 218, Exercise 5-1). Consider map $\Phi : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$\Phi(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y).$$

Show that $(0, 1) \in \mathbb{R}^2$ is a regular value of Φ, and the level set $\Phi^{-1}((0, 1))$ is diffeomorphic to 2-sphere S^2.

Exercise 3. **(GTM 218, Exercise 4-6)**. Let N be a non-empty smooth compact manifold. Show that there is no smooth submersion $F : N \to \mathbb{R}^k$ for any $k > 0$.

Exercise 4 (**A famous exercise**). Let $N \subset \mathbb{R}^m$ be a smooth submanifold of dimension $n \leq m - 3$. Prove that the complement $\mathbb{R}^m \backslash N$ is connected and simply connected.

Exercise 5 (**Differential Manifold Course from Prof. Zuoqin Wang, 2023, (4) in Problem set 3, Part 2**). Let $F : M \to M$ be a smooth map. A fixed point $p \in F$ (i.e., $F(p) = p$) is called non-degenerate if 1 is *not* an eigenvalue of the pushforward $F_*(p) : T_pM \to T_pM$. The map F is called a *Lefschetz map* if all its points are non-degenerate.

(1) Prove that the "horizontal" rotation $r_\theta : S^2 \to S^2$ by angle $\theta(\neq 2k\pi$ for any $k \in \mathbb{N})$ defined by

$$r_\theta(x, y, z) = (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta), z)$$

is a Lefschetz map, where S^2 here is viewed as a submanifold in $\mathbb{R}^3$ defined by $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$.

(2) Let V a vector space and $F : V \to V$ a linear map. Recall $\Delta = \{(v, v) \in V \times V \,|\, v \in V\}$ be the diagonal of $V \times V$ and $\Gamma_F = \{(v, F(v)) \in V \times V \,|\, v \in V\}$ be the graph of F on V. Prove that Γ_F intersects Δ transversally if and only if 1 is not an eigenvalue of F. Then deduce that if M is a compact manifold and $F : M \to M$ is a Lefschetz map, the there are only finitely many fixed points.

(3) When M is a compact manifold and $F : M \to M$ be a Lefschetz map, denote by

$$L(F) := \sum_{\text{fixed point } p \text{ of } F} \text{sign}(\det(F_*(p) - \mathbb{1}).$$

Here, sign means that if $\det(F_*(p) - \mathbb{1}) > 0$, then sign $= +1$ and if $\det(F_*(p) - \mathbb{1}) < 0$, then sign $= -1$. This $L(F)$ is a well-defined number and is called the *Lefschetz number* of Lefschetz map F. Compute $L(r_\theta)$ in Question (1) above.

Exercise 6. Recall that the group of $2n$-dimensional symplectic matrices is denoted by

$$\text{Sp}(2n) = \{A \in M_{2n\times 2n}(\mathbb{R}) \,|\, AJ_0A^{\text{T}} = J_0\}$$

where $J_0 \in M_{2n\times 2n}$ is defined by

$$J_0 = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix}.$$

Prove that $\text{Sp}(2n)$ is a submanifold of $M_{2n\times 2n}(\mathbb{R})$. Moreover, compute its dimension.

Exercise 7. Prove BY DEFINITION that if $N_1 \subset \mathbb{R}^{m_1}$ and $N_2 \subset \mathbb{R}^{m_2}$ are submanifolds of dimensions n_1 and n_2 respectively, then $N_1 \times N_2$ is a submanifold (of

$\mathbb{R}^{m_1+m_2}$) of dimension $n_1 + n_2$.

Exercise 8. (1) Prove the Inverse Mapping Theorem: Let $F : N \to M$ be a smooth map such that $F_*(p) : T_pN \to T_{F(p)}M$ is an isomorphism, then F is a diffeomorphism locally near p. (Hint: You may need the following Inverse Function Theorem in the Euclidean space: for open subset $U, V \subset \mathbb{R}^n$ and smooth map $f : U \to V$, if $dF(x)$ is an isomorphism for $x \in U$, then F is a diffeomorphism locally near x.) (2) Deduce from (1) that there is no immersion from S^n to $\mathbb{R}^n$.

Exercise 9. Let $F : N \to M$ be a smooth map. Recall that pullback of F is a functor $F^* : TM \to TN$. In particular, F^* defines a map for sections (forms) from $\Omega^k(M)$ to $\Omega^k(N)$ for any $k \in \mathbb{N}$, defined explicitly as follows,

$$(F^*\alpha)(X_1, ..., X_k) := \alpha(F_*(X_1), ..., F_*(X_k))$$

or even more explicitly when the positions are specified,

$$(F^*\alpha)(p)(X_1(p), ..., X_k(p)) := \alpha(F(p))(F_*(p)(X_1(p)), ..., F_*(p)(X_k(p))).$$

Now, consider map $F : \mathbb{R}^2 \to \mathbb{R}^3$, where $\mathbb{R}^2$ is in coordinate (x, y) and $\mathbb{R}^3$ in coordinate (u, v, w), by $F(x, y) = (xy, x^2, 3x + y)$. For $\alpha = uvdu + 2wdv - vdw \in \Omega^1(\mathbb{R}^3)$, compute $F^*\alpha$ and express it in terms of dx and dy.

Exercise 10 (GTM 218, Exercise 6-9). For map $F : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$F(x, y) = (e^y \cos x, e^y \sin x, e^{-y}).$$

Denote by $S_r(0) \subset \mathbb{R}^3$ the standard 2-sphere centered at 0 with radius r. Recall/Define that a map $F : N \to M$ is transitive to a submanifold $S \subset M$ means for any $x \in F^{-1}(S)$, linear spaces $T_{F(x)}S$ and $F_*(x)(T_xN)$ span $T_{F(x)}M$. (1) For which positive number r is F transitive to $S_r(0)$ in $\mathbb{R}^3$? (2) For which positive number r is $F^{-1}(S_r(0))$ an embedded submanifold of $\mathbb{R}^2$?