HOMEWORK FOR LECTURE 3

This homework problem set can be accomplished with the help of references. Every problem worths 1 point and DO NOT LEAVE ANY PROBLEM BLANK! It is due to 11:59 pm on November 5 (sharp).

Exercise 1. Let V be a vector space with basis $\{e_1, ..., e_n\}$. Then for a fixed $k \in \{1, ..., n\}$, prove that

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \,|\, 1 \leq i_1 < \cdots < i_k \leq n\}$$

form a basis of $\bigwedge^k V$. Therefore, $\dim \bigwedge^k V = \frac{n!}{k!(n-k)!}$.

Exercise 2. Let V be a vector space with basis $\{e_1, ..., e_n\}$, equipped with an inner product $\langle \cdot, \cdot \rangle$ with signature

$$(\underbrace{-, \cdots, -}_{p}, \underbrace{+, \cdots +}_{n-p}).$$

Prove that for the Hodge star operator $* : \bigwedge^k V^* \to \bigwedge^{n-k} V^*$, it satisfies

$$* \circ * = (-1)^{k(n-k)+p} \cdot \mathbb{1}_{\bigwedge^k V^*}$$

for any $k \in \{1, ..., n\}$.

Exercise 3. Let $\{\varphi_{s,t}\}_{(s,t)\in\mathbb{R}^2}$ be a 2-parametrized group of diffeomorphisms (on a manifold M). Consider two vector fields defined via the following equations,

$$\frac{\partial \varphi_{s,t}}{\partial t} = X_s \circ \varphi_{s,t} \quad \text{and} \quad \frac{\partial \varphi_{s,t}}{\partial s} = Y_t \circ \varphi_{s,t}.$$

Then prove the following equality,

$$\frac{\partial X_s}{\partial s} - \frac{\partial Y_t}{\partial t} = [X_s, Y_t]$$

where $[\cdot, \cdot]$ denotes the Poisson bracket of vector fields.

Exercise 4. Prove that, for vector fields X, Y (on a manifold M), the Lie derivative satisfies $\mathcal{L}_X Y = [X, Y]$.

Exercise 5. Recall that given a non-degenerate 2-form ω on M, any function $H : M \to \mathbb{R}$ corresponds to a vector field X_H defined by $-dH = \omega(X_H, \cdot)$. For two functions $H, G : M \to \mathbb{R}$, define

$$\{H, G\} := \omega(X_H, H_G).$$

Then prove that if ω is closed, i.e., $d\omega = 0$, then $\{\cdot, \cdot\}$ satisfies the Jacobi identity:

$$\{\{H, G\}, F\} + \{\{G, F\}, H\} + \{\{F, H\}, G\} = 0$$

for any functions $H, G, F : M \to \mathbb{R}$.

Exercise 6. **(GTM 218, Exercise 8-10)** Consider manifold $\mathbb{R}^2_{>0}$ and $\varphi : \mathbb{R}^2_{>0} \to \mathbb{R}^2_{>0}$ defined by

$$F(x, y) = \left(xy, \frac{y}{x}\right).$$

Compute the pushfoward $\varphi_* X$ for vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. Do the same thing for vector field $Y = y\frac{\partial}{\partial x}$.

Exercise 7. **(Partially based on GTM 218, Exercise 14-7 (a))** Consider 1-form $\alpha = xdy$ on $\mathbb{R}^2$ and map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\varphi(x, y) = (xy, e^{-y}).$$

Compute the pullback $\varphi^*\alpha$. Also, verify in this concrete case that $\varphi^*(d\alpha) = d(\varphi^*\alpha)$.

Exercise 8. **(Partially based on GTM 218, Exercise 9-17)** Let X be a smooth vector field on M^n such that $X(p) \neq 0$ at some point $p \in M$. (1) Prove that there exists a local chart $(U, \varphi : U \to V)$ near p, where V is an open subset of $\mathbb{R}^n$ in coordinates $(x_1, ..., x_n)$, such that within U, we have $\varphi_*(X) = \frac{\partial}{\partial x_1}$. (2) Given the following three vector fields on $\mathbb{R}^3$,

$$X_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad X_2 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad X_3 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

Near $p = (1, 0, 0)$, is it possible to find a local chart as above such that X_i maps to $\frac{\partial}{\partial x_i}$ for $i = 1, 2, 3$ at the same time? If so, construct such a local chart; if not, please give a justifying reason.

Exercise 9. Consider $\mathbb{R}^3$ equipped with the metric $g = dx \otimes dx + dy \otimes dy - dz \otimes dz$. A **Killing vector field** on $(\mathbb{R}^3, g)$ is a complete non-trivial vector field X such that

$\mathcal{L}_X g = 0$. In other words, by the definition of a Lie derivative, the flow generated by X preserves the metric g. (1) List as many *linearly independent* Killing vector fields in $(\mathbb{R}^3, g)$ as possible. (2) Verify that if X, Y are two Killing vector fields in $(\mathbb{R}^3, g)$, then $[X, Y]$ is also a Killing vector field in $(\mathbb{R}^3, g)$.

Exercise 10. Let α be a 1-form on M^3 satisfying $\alpha \wedge d\alpha$ is a nowhere vanishing 3-form on M^3. (1) Prove that there exists a vector field (called a **Reeb vector field**) denoted by R_α such that $d\alpha(R_\alpha, -) = 0$ and $\alpha(R_\alpha) = 1$. (2) Confirm that $\mathcal{L}_{R_\alpha}\alpha = 0$. (3) In $\mathbb{R}^3$ in coordinates (x, y, z), given an example of such α and work out the associated R_α.