HOMEWORK FOR LECTURE 2

This homework problem set can be accomplished with the help of references. Every problem worths 1 point and DO NOT LEAVE ANY PROBLEM BLANK! It is due to 11:59 pm on October 22 (sharp).

Exercise 1. Given the Grassmannian $\mathrm{Gr}_{\mathbb{R}}(k,n)$, consider the following set

$$\gamma_{\mathbb{R}}(k,n) := \{(V,v) \in \mathrm{Gr}_{\mathbb{R}}(k,n) \times \mathbb{R}^n \,|\, v \in V\}.$$

Prove that under the natural projection $\pi(V,v) := V$, the structure $\pi : \gamma_{\mathbb{R}}(k,n) \to \mathrm{Gr}_{\mathbb{R}}(k,n)$ is a real vector bundle of rank-k. This vector bundle is called the tautological bundle (over $\mathrm{Gr}_{\mathbb{R}}(k,n)$).

Exercise 2. Let X, Y be vector fields on M, and locally (within some (U_α, ϕ_α)) write $X = (X_1, ..., X_n)$ and $Y = (Y_1, ..., Y_n)$ where X_i, Y_j are smooth functions on U_α for $1 \leq i, j \leq n$. Prove that the Lie bracket locally writes as follows,

$$[X,Y] = (D_X Y_1 - D_Y X_1, \cdots, D_X Y_n - D_Y X_n).$$

Use this to calculate $[X,Y]$ for $X, Y \in \Gamma(T\mathbb{R}^3)$ (in coordinate (x,y,z)) where

$$X((x,y,z)) = (-y,x,0) \text{ and } Y((x,y,z)) = (0,-z,y).$$

Exercise 3. (1) Let $\mathbb{T}^2$ denote the 2-dimensional torus $S^1 \times S^1$. Construct a vector field $X \in \Gamma(T\mathbb{T}^2)$ that does *not* have any zero's. (2) Construct a vector field $X \in \Gamma(TS^2)$ that has only one zero.

Exercise 4. On the standard unit sphere S^3 in $\mathbb{R}^4$, construct three smooth vector fields $X, Y, Z \in \Gamma(TS^3)$ such that for every $p \in S^3$, the vectors $\{X(p), Y(p), Z(p)\}$ forms a basis at the fiber $T_pS^3 = \pi^{-1}(p)$ of the tangent bundle $\pi : TS^3 \to S^3$.

Exercise 5. Prove that for any finite-dimensional vector spaces U, V, W, there exists a map $\varphi : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ that is an isomorphism and identifies $u \otimes (v \otimes w)$ and $(u \otimes v) \otimes w$. (Hint: apply the universal property of tensor product.)

Exercise 6. Recall that an element $x \in V \otimes W$ is called *decomposable* if there exist $v \in V$ and $w \in W$ such that $x = v \otimes w$. Suppose V admits a basis $\{e_1, ..., e_n\}$ and W admits a basis $\{f_1, ..., f_m\}$. Prove that $x = \sum a_{ij}(e_i \otimes f_j) \in V \otimes W$ is decomposable if and only if the matrix $(a_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$ has rank 1.

Exercise 7. For any matrices $A \in GL(k, \mathbb{R})$ and $B \in GL(l, \mathbb{R})$, prove

$$\det(A \otimes B) = (\det(A))^k (\det(B))^l.$$

Exercise 8. Recall that on an even-dimensional manifold M, an *almost complex structure* denoted by J is a smooth family of morphism $J_x : T_xM \to T_xM$ satisfying $J_x^2 = -\mathbb{1}$. Consider the following $(1,2)$-tensor field

$$N_J(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

for any $X, Y \in \Gamma(TM)$. A celebrated result from Newlander–Nirenberg says that J is integrable (induced by a complex structure) if and only if $N_J \equiv 0$. Prove that over a closed surface Σ, any almost complex J (if exists) is always integrable.

Exercise 9. Prove that on any Riemannian manifold (M, g), there exists a unique connection ∇ satisfying, for any $X, Y, Z \in \Gamma(TM)$,

(i) (compatibility) $D_Z g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$
(ii) (torsion-free) $[X, Y] = \nabla_X Y - \nabla_Y X$.

Exercise 10. Given a Riemannian manifold (M, g), prove that for any smooth function $F : M \to \mathbb{R}$, there exists a unique vector field denoted by ∇F satisfying

$$g(\nabla F, X) = D_X F$$

for any $X \in \Gamma(TM)$. This vector field is called the *gradient of F on M*. Also, prove that function F is non-decreasing along ∇F. Finally, work out (with details) the explicit formula of ∇F for $F : (\mathbb{R}^2, g) \to \mathbb{R}$ in polar coordinate (r, θ), where g is taken as the standard inner product.