HOMEWORK FOR LECTURE 1

This homework problem set can be accomplished with the help of references. Every problem worths 1 point. It is due to October 8 (sharp).

Exercise 1. Prove that, for $1 \leq k \leq n$, the Grassmannian

$$\mathrm{Gr}_{\mathbb{R}}(k,n) = \{k\text{-dimensional linear subspaces of } \mathbb{R}^n\}$$

is a smooth manifold, by explicitly constructing open cover $\{U_\alpha\}_{\alpha \in I}$ and local charts $\{\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^{k(n-k)}\}_{\alpha \in I}$.

Exercise 2. Let M be a smooth manifold and $\phi \in \mathrm{Diff}(M)$. Prove that its graph $\mathrm{graph}(\phi) := \{(x, \phi(x)) \,|\, x \in M\}$ is a smooth manifold.

Exercise 3. Let M be a closed smooth manifold and $\phi \in \mathrm{Diff}(M)$. Prove that the mapping torus defined by

$$T_\phi(M) := [0,1] \times M/\sim$$

is a smooth manifold, where $(0,x)$ is identified with $(1, \phi(x))$ for any $x \in M$.

Exercise 4. Prove that the following set of matrices

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}$$

is a Lie group. Here "H" stands for Heisenberg.

Exercise 5. Assume the orthogonal group $O(n) = \{A \in M_{n\times n}(\mathbb{R}) \,|\, AA^T = \mathbb{1}\}$ is a compact Lie group of dimension $\frac{1}{2}n(n-1)$. Prove that the special orthogonal group

$$SO(n) := \{A \in O(n) \,|\, \det(A) = 1\}$$

is a compact Lie group and calculate its dimension.

Exercise 6. Prove that $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$ as two smooth manifolds.

Exercise 7. Identify $\mathbb{C}P^n$ with the set of equivalence classes in $(\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^*$. Consider the map $S : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^3$ by

$$([(w_0, w_1)], [(z_0, z_1)]) \mapsto [(w_0z_0, w_0z_1, w_1z_0, w_1z_1)].$$

Prove that S is a smooth map. Here, "S" stands for Segre.

Exercise 8. Consider group $E(n) := \mathbb{R}^n \rtimes O(n)$ where the multiplication is given by

$$(v, A) \cdot (w, B) = (v + Aw, AB)$$

where "E" stands for Euclidean. Note that $E(n)$ is a Lie group. Meanwhile, a representation of $E(n)$ is a Lie group homomorphism from $E(n)$ to $GL(k, \mathbb{R})$ for some $k > 0$. Construct a non-trivial representation of $E(n)$ that is injective.

Exercise 9. Prove that the upper half-plane in $\mathbb{C}$, denoted by

$$\mathbb{H} := \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$$

is a homogenous space.

Exercise 10. Prove that if M and N are smooth diffeomorphic, then $\dim M = \dim N$.