Ex 1:

(1).

$\Leftarrow$: Obvious:

$\Rightarrow$: If $\Delta f = 0$. $\quad 0 = (\Delta f, f) = (df, df) \geq 0$. $\quad \Rightarrow df \equiv 0$ on M.

$\Rightarrow f$ = constant.

If $\Delta(f\Omega) = 0$. $\quad 0 = (f\Omega, \Delta(f\Omega)) = (\delta(f\Omega), \delta(f\Omega)) \geq 0$.

$\Rightarrow \delta(f\Omega) = 0$. $\quad (-1)^{n(n-1)+1} * d * (f\Omega) = 0 \quad *df = 0 \quad \Rightarrow \quad df = 0$.

$\Rightarrow f$ = constant.

(2).

$\Leftarrow$: $\int_M f\Omega = \int_M \Delta g \Omega = (\Delta g, 1) = (g, \Delta 1) = 0$.

$\Rightarrow$: If $\int_M f\Omega = 0$. $\exists$ $(n-1)$-form η. s.t. $f\Omega = d\eta$.

From Hodge Theorem:

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus \mathrm{Im}(\Delta : \Omega^k(M) \to \Omega^k(M))$$
$$= \mathcal{H}^k(M) \oplus \mathrm{Im}(d\delta) \oplus \mathrm{Im}(\delta d).$$
$$= \mathcal{H}^k(M) \oplus \mathrm{Im}(d) \oplus \mathrm{Im}(\delta)$$

$\exists$ n-form τ. s.t.

$$f\Omega = d\eta = d\delta\tau = \Delta\tau.$$

$\tau = g\Omega$. $\quad \Rightarrow \Delta\tau = (\Delta g)\Omega = f\Omega$.

$f = \Delta g$.

Ex 2:

(1). $\forall X, Y \in D^2$. $\forall p \in M$, $\alpha(X) = 0$, $\alpha(Y) = 0$

$$d\alpha(X, Y) = Y\alpha(X) - X\alpha(Y) - \alpha([X, Y]).$$
$$= -\alpha([X, Y]) \neq 0 \quad \text{since } \alpha \wedge d\alpha \neq 0.$$

$\Rightarrow [X, Y] \notin \operatorname{Ker}\alpha = D^2$.

(2).

$X_1 = (0, 0, 1)$. $X_2 = (\cos(2\pi z), -\sin(2\pi z), 0)$.

Define $\alpha = \sin(2\pi z)dx + \cos(2\pi z)dy$, $\operatorname{Ker}\alpha = D^2$.

$$d\alpha = 2\pi\cos(2\pi z)dz \wedge dx - 2\pi\sin(2\pi z)dz \wedge dy.$$
$$d\alpha \wedge \alpha = 2\pi\, dx \wedge dy \wedge dz \neq 0.$$

(3). $\gamma(t) = (\frac{1}{2\pi}\sin(2\pi t), \frac{1}{2\pi}\cos(2\pi t), t)$. $t \in [0, 1)$.

$$\gamma'(t) = \cos(2\pi t)\frac{\partial}{\partial x} - \sin(2\pi t)\frac{\partial}{\partial y} + \frac{\partial}{\partial z} = X_1 + X_2$$

Ex 3:

Let $f: S^1 \to S^n$ be a smooth map. Sard's theorem $\Rightarrow \exists\, p \in S^n$, p is a regular value of f. Let $\pi: S^n \setminus \{p\} \to \mathbb{R}^n$ be the stereographic projection around p.

If there is an $x \in S^1$ s.t. $p = f(x) \Rightarrow df_x: T_xS^1 \to T_pS^n$ is a map which can not be surjective. $\Rightarrow p \notin \operatorname{im} f$.

$\pi \circ f: S^1 \to S^n \setminus \{p\} \to \mathbb{R}^n$ is null-homotopic since $\mathbb{R}^n$ is contractible. $\Rightarrow f$ is null-homotopic.

$\Rightarrow S^n$ is simply-connected.