2024秋 微分流形 第7次作业答案

EXERCISE 1

Pf. Denote: $0 \to C^p \xrightarrow{f^p} D^p \xrightarrow{g^p} E^p \to 0$

f^p, g^p : chain mappings, i.e. $\begin{cases} d_D^p \circ f^p = f^{p+1} \circ d_C^p, \\ d_E^p \circ g^p = g^{p+1} \circ d_D^p \end{cases}$

so: if $[\alpha] \in H^p(C;\mathbb{K})$, $\longrightarrow [f^p\alpha] \in H^p(D;\mathbb{K})$

$d_C^p \alpha = 0 \Rightarrow (d_D^p \circ f^p)\alpha = f^{p+1}(d_C^p \alpha) = 0;$

$\alpha = d_C^{p-1}\beta \Rightarrow f^p\alpha = d_D^{p-1}(f^{p-1}\beta)$

same as $H^p(D;\mathbb{K})$, so $\begin{cases} \tilde{f}^p : H^p(C;\mathbb{K}) \to H^p(D;\mathbb{K}) \\ \tilde{g}^p : H^p(D;\mathbb{K}) \to H^p(E;\mathbb{K}) \end{cases}$ are

both well-defined

Let: $H^p(C;\mathbb{K}) \xrightarrow{\tilde{f}^p} H^p(D;\mathbb{K}) \xrightarrow{\tilde{g}^p} H^p(E;\mathbb{K}) \xrightarrow{\tilde{h}^p} H^{p+1}(C;\mathbb{K})$

(i) $\ker \tilde{g}^p = \operatorname{Im} \tilde{f}^p$. Easy to check: $\tilde{g}^p \circ \tilde{f}^p = 0$, since

$0 \to C^p \xrightarrow{f^p} D^p \xrightarrow{g^p} E^p \to 0 \Rightarrow g^p \circ f^p = 0 \Rightarrow \ker \tilde{g}^p \supset \operatorname{Im} \tilde{f}^p$

$\Rightarrow$ Take $[\alpha] \in \ker \tilde{g}^p$, then $\exists \beta$, $g^p\alpha = d_E^{p-1}\beta$.

Since g^p is surjective, $\exists \delta$ s.t. $\beta = g^{p-1}\delta$

Then $\alpha - d_D^{p-1}\delta \in \ker g^p = \operatorname{Im} f^p$, $\exists \theta$ s.t. $\alpha - d_D^{p-1}\delta = f^p\theta$

$\Rightarrow [\alpha] = [f^p\theta] = \tilde{f}^p[\theta] \Rightarrow$ # ✓

(ii) $\ker \tilde{h}^p = \operatorname{Im} \tilde{g}^p$.

<u>Recall</u> the construction of $\tilde{h}^p$:

$$\begin{tikzcd} & D^p \arrow[r, "g^p"] \arrow[d, "d_D^p"'] & E^p \arrow[d, "d_E^p"] \\ C^{p+1} \arrow[r, "f^{p+1}"] & D^{p+1} \arrow[r, "g^{p+1}"] & E^{p+1} \end{tikzcd}$$

Take $[e] \in H^p(E; \mathbb{K})$

$\exists \alpha \in D^p$, $g^p \alpha = e$.

$g^{p+1} \circ d_D^p \alpha = d_E^p \circ g^p \alpha = d_E^p e = 0$

$\Rightarrow d_D^p \alpha \in \operatorname{Im} f^{p+1}$

$\exists \beta \in C^{p+1}$, $f^{p+1} \beta = d_D^p \alpha$

$\tilde{h}^p : [e] \mapsto [\beta]$. (Note $d_C^{p+1} \beta = 0$, since: f^{p+1} injective)

$d_D^{p+1} \circ f^{p+1} \beta = 0 = f^{p+1} \circ d_C^p \beta = 0$

① $[\beta] = 0$, then $\beta = d_C^p \theta$, $d_D^p \alpha = f^{p+1} \circ d_C^p \theta = d_D^p \circ f^p \theta$

$\alpha - f^p \theta \in \ker d_D^p$. Since $g^p \circ f^p = 0$, then: $e = g^p \alpha =$

$g^p(\alpha - f^p \theta) \Rightarrow [e] = \tilde{g}^p [\alpha - f^p \theta] \Rightarrow \ker \tilde{h}^p \subset \operatorname{Im} \tilde{g}^p$.

② $[e] = \tilde{g}^p [\theta]$, one may take $\alpha = \theta$

$\Rightarrow f^{p+1} \beta = d_D^p \alpha = 0 \Rightarrow \beta = 0 \Rightarrow \operatorname{Im} \tilde{g}^p \subset \ker \tilde{h}^p$.

(iii) $\ker \tilde{f}^{p+1} = \operatorname{Im} \tilde{h}^p$.

① $\tilde{h}^p : [e] \mapsto [\beta]$, $\tilde{f}^{p+1}[\beta] = [f^{p+1} \beta] = [d_D^p \alpha] = 0$

$\Rightarrow \tilde{f}^{p+1} \circ \tilde{h}^p = 0 \Rightarrow \operatorname{Im} \tilde{h}^p \subset \ker \tilde{f}^{p+1}$;

② $\tilde{f}^{p+1}[\beta] = 0 = [f^{p+1}\beta]$, then $\exists\, \tilde{\beta}$, $f^{p+1}\beta = d_D^p \tilde{\beta}$.

$g^{p+1} \circ f^{p+1}\beta = 0 = g^{p+1} \circ d_D^p \tilde{\beta} = d_E^p \circ g^p \tilde{\beta}$

$$\begin{array}{ccc} \tilde{\beta} \in D^p & \xrightarrow{g^p} & E^p \ni g^p\tilde{\beta} \\ \downarrow & & \\ \beta \in C^{p+1} \xrightarrow{f^{p+1}} & D^{p+1} \ni d_D^p \tilde{\beta} & \end{array}$$

definition of $\tilde{h}^p$:

$\tilde{h}^p[g^p\tilde{\beta}] = [\beta]$

$\Rightarrow \ker \tilde{f}^{p+1} \subset \operatorname{Im} \tilde{h}^p$.

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EXERCISE 2 (KÜNNETH FORMULA)

<u>Rmk</u> Necessary condition for Künneth formula:

$$\dim_{\mathbb{R}} H^k_{dR}(M;\mathbb{R}) < +\infty, \ \forall k.$$

Counterexample happens when $H^{\cdot}_{dR}(M;\mathbb{R})$ is not finitely generated.

<u>Pf</u> (i) HOMOLOGICAL ALGEBRA APPROACH:

<u>c.f.</u> J. Munkres, ELEMENTS OF ALGEBRAIC TOPOLOGY

KÜNNETH'S THEOREM: (Ch 7, P353)

Let X, Y be topological spaces, suppose $H_i(X)$ is finitely generated for each i. Then:

$$0 \to \bigoplus_{p+q=m} H^p(X) \otimes H^q(Y) \to H^m(X \times Y)$$

$$\to \bigoplus_{p+q=m} \underbrace{H^{p+1}(X) * H^q(Y)}_{\text{Torsion part}} \to 0.$$

$\Rightarrow$ The case for de Rham: coeiffcients are in field $\mathbb{R}$, torsion parts vanish. $\Rightarrow$ Künneth. #

(ii) MAYER-VIETORIS SEQUENCE & GOOD COVER

c.f. Prof Zuoqing Wang's Lecture Notes, 2023
Lec 24. P168.

Mainly by induction on the number of good cover on M. Let $\begin{array}{l}\pi_1 : M\times N \to M \\ \pi_2 : M\times N \to N\end{array}$ projections,

$$\Psi : \Omega^i(M)\otimes\Omega^j(N) \to \Omega^i(M\times N),$$
$$\omega_1\otimes\omega_2 \longmapsto \pi_1^*\omega_1\wedge\pi_2^*\omega_2$$

induced $\Psi : H^{\cdot}_{dR}(M;\mathbb{R})\otimes H^{\cdot}_{dR}(N;\mathbb{R}) \to H^{\cdot}_{dR}(M\times N;\mathbb{R})$,

$[\omega_1]\otimes[\omega_2] \longmapsto [\pi_1^*\omega_1\wedge\pi_2^*\omega_2]$. (Let $\tilde{H}^k(M,N) = \bigoplus_{p+q=k} H^p(M)\otimes H^q(N)$)

<u>Goal</u>: Ψ is a isomorphism. (☆)

Induction: $M = \underbrace{U_1\cup\cdots\cup U_\ell}_{\text{good cover}}$, suppose ☆ holds $\tilde{M} = U_1\cup\cdots\cup U_s$

when $\ell = s$, then for $\ell = s+1$: check:

M-V $\Rightarrow$
$$\begin{array}{ccccccccc}
\tilde{H}^k(M,N) & \to & \tilde{H}^k(\tilde{M};N)\oplus\tilde{H}^k(U_{s+1};N) & \to & \tilde{H}^k(\tilde{M}\cap U_{s+1};N) & \to & \tilde{H}^{k+1}(M;N) \\
\Psi\downarrow & & \Psi\downarrow & & \Psi\downarrow & & \downarrow\Psi \\
H^k_{dR}(M\times N) & \to & H^k_{dR}(\tilde{M}\times N)\oplus H^k_{dR}(U_{s+1}\times N) & \to & H^k_{dR}((\tilde{M}\cap U_{s+1})\times N) & \to & H^{k+1}_{dR}(M\times N)
\end{array}$$

five Lemma $\Rightarrow$ #

EXERCISE 3

Sol Recall $\mathbb{RP}^n \simeq \mathbb{S}^n/\sim$, where $x \sim -x$.

Let $\pi: \mathbb{S}^n \to \mathbb{RP}^n$ be the covering map, then:

Take $U_1 := \{(x^1, \cdots, x^{n+1}) \in \mathbb{S}^n : x^{n+1} > \frac{1}{3}\}$, $U_2 := \{(x^1, \cdots, x^{n+1}) \in \mathbb{S}^n : |x^{n+1}| < \frac{1}{2}\}$, $\Big\} \overset{\text{open}}{\subseteq} \mathbb{S}^n$,

$V_1 := \pi(U_1)$, $V_2 := \pi(U_2)$, $V_1, V_2 \overset{\text{open}}{\subseteq} \mathbb{RP}^n$ since:

$\pi^{-1}(V_1) = U_1 \cup \{(x^1, \cdots, x^{n+1}) \in \mathbb{S}^n : x^{n+1} < -\frac{1}{3}\} \overset{\text{open}}{\subseteq} \mathbb{S}^n$,

$\pi^{-1}(V_2) = U_2 \overset{\text{open}}{\subseteq} \mathbb{S}^n$

& $V_1 \cap V_2 = \pi(U_1 \cap U_2)$ homotopic to $\mathbb{S}^{n-1}$, & V_2 is homotopic to $\mathbb{RP}^{n-1}$ (since U_2 is homotopic to $\mathbb{S}^{n-1}$),

$$\overset{\text{MVseq}}{\Longrightarrow} \cdots \to H^k_{dR}(\mathbb{RP}^n; \mathbb{R}) \xrightarrow{\alpha_k} H^k_{dR}(V_1; \mathbb{R}) \oplus H^k_{dR}(V_2; \mathbb{R})$$
$$\xrightarrow{\beta_k} H^k_{dR}(V_1 \cap V_2; \mathbb{R}) \xrightarrow{\delta_k} H^{k+1}_{dR}(\mathbb{RP}^n; \mathbb{R}) \to \cdots$$

where $H^k_{dR}(V_1; \mathbb{R}) \simeq \begin{cases} \mathbb{R}, & k=0; \\ 0, & k>0. \end{cases}$ as V_1 is homeomorphic to $\mathbb{R}^n$;

$H^k_{dR}(V_2; \mathbb{R}) \simeq H^k_{dR}(\mathbb{RP}^{n-1}; \mathbb{R})$, $H^k_{dR}(V_1 \cap V_2; \mathbb{R}) \simeq H^k_{dR}(\mathbb{S}^{n-1}; \mathbb{R})$

& $H^k_{dR}(\mathbb{S}^{n-1}; \mathbb{R}) = \begin{cases} \mathbb{R}, & k=0, n-1, \\ 0, & \text{else}, \end{cases}$

$\underline{\text{FACT}}$ $H^n_{dR}(\mathbb{RP}^n;\mathbb{R}) = \begin{cases} \mathbb{R}, & n=2k+1, k\in\mathbb{N}, \\ 0, & n=2k, k\in\mathbb{N}_+. \end{cases}$

when $1<k<n-1$ $(n>2)$:

$$0 \xrightarrow{\delta_{k-1}} H^k_{dR}(\mathbb{RP}^n;\mathbb{R}) \xrightarrow{\alpha_k} H^k_{dR}(\mathbb{RP}^{n-1};\mathbb{R}) \xrightarrow{\beta_k} 0$$

so $H^k_{dR}(\mathbb{RP}^n;\mathbb{R}) \simeq H^k_{dR}(\mathbb{RP}^{n-1};\mathbb{R})$, $1<k<n-1$;

when $k=n-1$ $(n>2)$:

$$0 \xrightarrow{\delta_{n-2}} H^{n-1}_{dR}(\mathbb{RP}^n;\mathbb{R}) \xrightarrow{\alpha_{n-1}} H^{n-1}_{dR}(\mathbb{RP}^{n-1};\mathbb{R})$$
$$\xrightarrow{\beta_{n-1}} H^{n-1}_{dR}(S^{n-1};\mathbb{R}) \simeq \mathbb{R}$$
$$\xrightarrow{\delta_{n-1}} H^n_{dR}(\mathbb{RP}^n;\mathbb{R}) \xrightarrow{\alpha_n} 0$$

$\Rightarrow$ when $n=2k$,

$$0 \xrightarrow{\delta_{2k-2}} H^{2k-1}_{dR}(\mathbb{RP}^{2k};\mathbb{R}) \xrightarrow{\alpha_{2k-1}} \underset{\underset{0}{\parallel}}{\mathbb{R}} \xrightarrow{\beta_{2k-1}} \mathbb{R} \xrightarrow{\delta_{2k-1}} 0$$

So: $H^{2k-1}_{dR}(\mathbb{RP}^{2k};\mathbb{R}) \simeq 0$;

surjective, so injective

$\Rightarrow \mathrm{Im}(\alpha_{2k-1}) = 0$

$\Rightarrow$ when $n=2k+1$,

$$0 \xrightarrow{\delta_{2k-1}} H^{2k}_{dR}(\mathbb{RP}^{2k+1};\mathbb{R}) \xrightarrow{\alpha_{2k}} H^{2k}_{dR}(\mathbb{RP}^{2k};\mathbb{R}) \simeq 0$$

So: $H^{2k}_{dR}(\mathbb{RP}^{2k+1};\mathbb{R}) \simeq 0$.

So: $H^{n-1}_{dR}(\mathbb{RP}^n;\mathbb{R}) \simeq 0$, $n>2$.

& when $1 < k < n-1$ $(n>2)$

$$H^k_{dR}(\mathbb{R}P^n;\mathbb{R}) \simeq \cdots \simeq H^k_{dR}(\mathbb{R}P^{k+1};\mathbb{R}) \simeq 0.$$

Since $\pi: \mathbb{S}^n \to \mathbb{R}P^n$ $(n>1)$ is a double cover,

so $\pi_1(\mathbb{R}P^n) \simeq \mathbb{Z}_2$.

<u>FACT</u> $H^1_{dR}(M;\mathbb{R}) \simeq \mathrm{Hom}(\pi_1(M):\mathbb{R})$

(See Allen Hatcher, Algebraic Topology)

$\Rightarrow H^1_{dR}(\mathbb{R}P^n;\mathbb{R}) \simeq 0.$

So:
$$H^k_{dR}(\mathbb{R}P^n;\mathbb{R}) \simeq \begin{cases} \mathbb{R}, & k=0,\ k=n \text{ odd}, \\ 0, & \text{else.} \end{cases}$$

EXERCISE 4.

pf. POINCARÉ'S DUALITY : $H^i_{dR}(M;\mathbb{R}) \simeq H^{k-i}_{dR}(M;\mathbb{R})$

M^k closed & oriented.

BETTI NUMBER : $b_i(M;\mathbb{R}) = \dim_{\mathbb{R}} H^i_{dR}(M;\mathbb{R})$.

So: $$\chi(M) = \sum_{i=0}^{4n+2} (-1)^i b_i(M;\mathbb{R})$$

$$= \sum_{i=0}^{2n} (-1)^i b_i(M;\mathbb{R}) + \sum_{i=2n+2}^{4n+2} (-1)^i b_i(M;\mathbb{R})$$

$$- b_{2n+1}(M;\mathbb{R}).$$

By PD, $b_i(M;\mathbb{R}) = b_{4n+2-i}(M;\mathbb{R})$, thus

$$\sum_{i=2n+2}^{4n+2} (-1)^i b_i(M;\mathbb{R})$$
$$= \sum_{i=2n+2}^{4n+2} (-1)^i b_{4n+2-i}(M;\mathbb{R})$$
$$= \sum_{i=0}^{2n} (-1)^{4n+2-i} b_i(M;\mathbb{R})$$
$$= \sum_{i=0}^{2n} (-1)^i b_i(M;\mathbb{R}),$$

and $$\chi(M) = \underbrace{2\sum_{i=0}^{2n} (-1)^i b_i(M;\mathbb{R})}_{\text{even number}} - b_{2n+1}(M;\mathbb{R}).$$

Recall: PD is given by:

$$H^{i}_{dR}(M;\mathbb{R}) \times H^{k-i}_{dR}(M;\mathbb{R}) \to \mathbb{R}$$

$$([\omega],[\theta]) \mapsto \int_M \omega\wedge\theta$$

So: $P: H^{2n+1}_{dR}(M;\mathbb{R}) \times H^{2n+1}_{dR}(M;\mathbb{R}) \to \mathbb{R}$,

$$P([\omega],[\theta]) = \int_M \omega\wedge\theta$$

is a non-degenerating anti-symmetric bilinear form over $H^{2n+1}_{dR}(M;\mathbb{R})$.

Standard linear algebra result: $\dim_{\mathbb{R}} H^{2n+1}_{dR}(M;\mathbb{R})$ is even.

So: $\chi(M) = \underbrace{2\sum_{i=0}^{2n}(-1)^i b_i(M;\mathbb{R})}_{\text{even}} - \underbrace{b_{2n+1}(M;\mathbb{R})}_{\text{even}}$ is even.

$\#$

EXERCISE 5.

(1) Sol. $\mathbb{T}^n \simeq \underbrace{S^1 \times \cdots \times S^1}_{n}$ closed, oriented.

Recall DE RHAM COHOMOLOGY OF S^1

$H^0_{dR}(S^1; \mathbb{R}) \simeq \mathbb{R}$, $H^1_{dR}(S^1; \mathbb{R}) \simeq \mathbb{R}$

Use unit complex numbers to represent S^1, i.e.

$S^1 \simeq \{e^{i\theta} \mid \theta \in \mathbb{R}\}$, $\underbrace{\text{volume form}}_{\text{generator of } H^1_{dR}(S^1;\mathbb{R})}$ of S^1 is $d\theta$

Volume form of $\mathbb{T}^n$: $d\theta_1 \wedge \cdots \wedge d\theta_n$

(each $d\theta_i$ is the volume form of the i-th S^1)

$f: (e^{i\theta_1}, \cdots, e^{i\theta_n}) \mapsto (e^{ik_1\theta_1}, \cdots, e^{ik_n\theta_n})$. So:

$$f^* d\theta_i = k_i \, d\theta_i,$$

$$f^*(d\theta_1 \wedge \cdots \wedge d\theta_n) = (f^* d\theta_1) \wedge \cdots \wedge (f^* d\theta_n)$$
$$= \left(\prod_{i=1}^{n} k_i\right) d\theta_1 \wedge \cdots \wedge d\theta_n$$

$$\Rightarrow \deg(f) = \prod_{i=1}^{n} k_i.$$

(2) <u>Pf</u> <u>Recall</u>: COHOMOLOGY WITH COEFFICIENT

$$H^k(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & k=2,4,\cdots,2n, \\ 0, & \text{else}; \end{cases}$$

$$H^k(S^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & k=0,n, \\ 0, & \text{else}. \end{cases}$$

CUP PRODUCT IN COHOMOLOGY

$$\cup : H^k(M;\mathbb{Z}) \times H^\ell(M;\mathbb{Z}) \to H^{k+\ell}(M;\mathbb{Z})$$

$\Rightarrow$ In de Rham cohomology, cup product is just the wedge of differential forms:

$$\wedge : H^k_{dR}(M;\mathbb{R}) \times H^\ell_{dR}(M;\mathbb{R}) \to H^{k+\ell}_{dR}(M;\mathbb{R})$$

$$([\alpha],[\beta]) \mapsto [\alpha]\wedge[\beta] := [\alpha\wedge\beta]$$

$\Rightarrow$ same properties of cup product:

If $f \in C^\infty(M,N)$, then $f^*: H^k(N;\mathbb{Z}) \to H^k(M;\mathbb{Z})$

& $f^*[\alpha] \cup f^*[\beta] = f^*([\alpha]\cup[\beta])$;

If $k+\ell > \dim M$, then: $\underset{H^k(M;\mathbb{Z})}{\underset{\cap}{[\alpha]}} \cup \underset{H^\ell(M;\mathbb{Z})}{\underset{\cap}{[\beta]}} = 0$

Cohomology Structure of $\mathbb{C}P^n$

$\Rightarrow$ $\mathbb{C}P^n$ is symplectic/ Kähler-Einstein, & $H^2_{dR}(\mathbb{C}P^n;\mathbb{R})$ was generated by its symplectic form ω; also, $\forall\, 1\leq k\leq n$, $\omega^k := \underbrace{\omega\wedge\cdots\wedge\omega}_{k} \in \Omega^{2k}(\mathbb{C}P^n)$ is nowhere vanishing $d\omega^k = 0$. So:

$$H^{2k}_{dR}(\mathbb{C}P^n;\mathbb{R}) = \mathrm{span}_{\mathbb{R}}\{[\omega^k]\}$$

$$= \mathrm{span}_{\mathbb{R}}\{\underbrace{[\omega]\wedge\cdots\wedge[\omega]}_{k}\} \simeq \mathbb{R},\quad 1\leq k\leq n.$$

$\Rightarrow$ Same for $H^{2k}(\mathbb{C}P^n;\mathbb{Z})$: Let $[\alpha]\in H^2(\mathbb{C}P^n;\mathbb{Z})\simeq\mathbb{Z}$ be a generator, then:

$$H^{2k}(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[\overbrace{[\alpha]\cup\cdots\cup[\alpha]}^{k}]$$

$$= \{n\,\overbrace{[\alpha]\cup\cdots\cup[\alpha]}^{k} : n\in\mathbb{Z}\} \simeq \mathbb{Z}.$$

Cohomology Structure of $S^2\times S^2$

$\Rightarrow$ de Rham case: Let θ_1, θ_2 be the volume form of S_1^2, S_2^2 relatively. $\begin{cases}\pi_1: S_1^2\times S_2^2\to S_1^2\\ \pi_2: S_1^2\times S_2^2\to S_2^2\end{cases}$ projection. then:

$$H^4_{dR}(S^2\times S^2;\mathbb{R}) \simeq \mathrm{span}_{\mathbb{R}}\{[\pi_1^*\theta_1\wedge\pi_2^*\theta_2]\} \simeq \mathbb{R},$$

$$H^2_{dR}(S^2\times S^2;\mathbb{R}) \simeq \mathrm{span}_{\mathbb{R}}\{[\pi_1^*\theta_1]\}\oplus\mathrm{span}_{\mathbb{R}}\{[\pi_2^*\theta_2]\}$$

$$\simeq \mathbb{R}^2,$$

$\Rightarrow$ Same for $H^k(S^2\times S^2;\mathbb{Z})$:

$[\alpha_1]\in H^2(S^2_1;\mathbb{Z})$
$[\alpha_2]\in H^2(S^2_2;\mathbb{Z})$
generators.

$$H^2(S^2\times S^2;\mathbb{Z})\simeq\mathbb{Z}[[\pi_1^*\alpha_1]]\oplus\mathbb{Z}[[\pi_2^*\alpha_2]]$$
$$\simeq\mathbb{Z}\oplus\mathbb{Z}\simeq\mathbb{Z}^2,$$
$$H^4(S^2\times S^2;\mathbb{Z})\simeq\mathbb{Z}[[\pi_1^*\alpha_1]\cup[\pi_2^*\alpha_2]]\simeq\mathbb{Z}.$$

Let $f: S^2\times S^2\to\mathbb{C}P^2$. then:

$$f^*: H^2(\mathbb{C}P^2;\mathbb{Z})\to H^2(S^2\times S^2;\mathbb{Z}),$$
$$f^*[\alpha]=k_1[\pi_1^*\alpha_1]+k_2[\pi_2^*\alpha_2].\quad k_1,k_2\in\mathbb{Z};$$

$$f^*: H^4(\mathbb{C}P^2;\mathbb{Z})\to H^4(S^2\times S^2;\mathbb{Z}),$$
$$f^*([\alpha]\cup[\alpha])=\deg(f)([\pi_1^*\alpha_1]\cup[\pi_2^*\alpha_2])$$

$[\alpha]\in H^k$
$[\beta]\in H^l$
$\Rightarrow[\alpha]\cup[\beta]$
$=(-1)^{kl}[\beta]\cup[\alpha]$

$$=(f^*[\alpha])\cup(f^*[\alpha])$$
$$=(k_1[\pi_1^*\alpha_1]+k_2[\pi_2^*\alpha_2])\cup$$
$$(k_1[\pi_1^*\alpha_1]+k_2[\pi_2^*\alpha_2])$$
$$=k_1^2\,\pi_1^*\underbrace{([\alpha_1]\cup[\alpha_1])}_{=0}+k_2^2\,\pi_2^*\underbrace{([\alpha_2]\cup[\alpha_2])}_{=0}$$
$$+2k_1k_2[\pi_1^*\alpha_1]\cup[\pi_2^*\alpha_2]$$

So $\deg(f)=2k_1k_2$ is even. #

<u>Rmk</u> Assume M^k, N^k oriented closed mfds, $f \in C^\infty(M,N)$.

$$f^*: H^k_{dR}(N;\mathbb{R}) \to H^k_{dR}(M;\mathbb{R})$$

$$\begin{array}{ccc} \wr & & \wr \\ \mathbb{R} & & \mathbb{R} \end{array} \qquad \int_M f^*\omega = \deg(f) \int_N \omega$$

generator: volume form ω $\qquad$ <u>FACT</u> $\deg(f) \in \mathbb{Z}$.

- $f: M \to N$ diffeomorphsim, then $\deg(f) = \pm 1$.

  $\deg(f) > 0$: orientation preserving. $\qquad$ (Sard)

- local degree; $\deg(f,p)$ for $p \in M$, $f(p) \notin f(\mathrm{Crit}(f))$

<u>Lem</u>: $\# f^{-1}(f(p)) < +\infty$. Let $N = \# f^{-1}(f(p))$

& $f^{-1}(f(p)) = \{p_i\}_{i=1}^N$, then:

$\exists\, U_i \subset M$ nbhd of p_i, $V \subset N$ nbhd of $f(p)$,

then: $f_i := f|_{U_i}: U_i \to V$ is homeomorphism.

$\Rightarrow \deg(f, p_i) = \deg(f_i) = \pm 1$.

$\Rightarrow \deg(f) = \sum_{i=1}^N \deg(f, p_i) \in \mathbb{Z}$.

More generally, for $q \notin f(\mathrm{Crit}(f))$, then

$$\deg(f) = \sum_{p \in f^{-1}(q)} \deg(f,p)$$

where $\deg(f,p) := \mathrm{sgn}(\det(df_p: T_pM \to T_qN))$. $(= \pm 1)$