Ex 1.

(i). (ii):

For $GL(n.F)$. $F=\mathbb{R}$ or $\mathbb{C}$. $SL(n.F)$ is closed subgroup of $GL(n.F)$. so it is Lie subgroups of $GL(n.F)$.

Let $sl(n.F)$ be the subspace of $gl(n.F)$ s.t.

$$sl(n.F) := \{ A \in gl(n.F) \mid \operatorname{trace}(A) = 0 \}.$$

For $B.C \in sl(n.F)$. $\quad Tr([B.C]) = Tr(BC-CB) = 0.$

$$\Rightarrow [B.C]_{gl(n.F)} \in sl(n.F)$$

$\Rightarrow$ $sl(n.F)$ is a Lie subalgebra.

The map $X \mapsto e^X$ gives a diffeomorphism

To end the proof. note that

$$\{ A \in gl(n.F) \mid \exp_{gl(n.F)}(tA) \in SL(n.F) \ \forall t \in \mathbb{R} \}.$$
$$= sl(n.F).$$

Indeed. $\quad e^{tA} \in SL(n.F) \Leftrightarrow \det(e^{tA}) = 1 \quad \forall t$

$$\Leftrightarrow Tr(tA) = 0 \quad \forall t$$

$$\Leftrightarrow Tr(A) = 0.$$

(iii). (iv).

$O(n) / U(n)$. is closed subgroup of $GL(n.\mathbb{R}) / GL(n.\mathbb{C})$. thus Lie subgroup. As Lie subalgebra:

$$\mathfrak{g}_{O(n)} = \{ A \in gl(n.\mathbb{R}) \mid \exp(tA) \in O(n) \ \forall t \in \mathbb{R} \}.$$

$$\exp(tA) \in O(n) \Leftrightarrow e^{tA}(e^{tA})^T = I$$

$$\Leftrightarrow \forall t: \quad I = (I + tA + \tfrac{1}{2}t^2A^2 + \cdots)(I + tA^T + \tfrac{1}{2}t^2 A^{T^2} + \cdots)$$

$$= I + t(A + A^T) + \tfrac{1}{2}t^2(A^2 + 2AA^T + A^{T^2}) + \cdots$$

$\Rightarrow$ Each coefficient in this series must vanish except I.

$$\Leftrightarrow \quad A + A^T = 0.$$

So

$$\mathfrak{g}_{O(n)} = o(n) = \{ A \in gl(n.\mathbb{R}) \mid A^T + A = 0 \}.$$

Similar case for

$$\mathfrak{g}_{U(n)} = \mathfrak{u}(n) = \{A \in \mathfrak{gl}(n, \mathbb{C}) \mid A^* + A = 0\}.$$

(V).

$$\mathfrak{g}_{Sp(2n)} = \{A \in \mathfrak{gl}(2n, \mathbb{R}) \mid \exp(tA) \in Sp(2n) \quad \forall t \in \mathbb{R}\}.$$

$$e^{tA} J (e^{tA})^T = J.$$

$$\Leftrightarrow \quad J e^{tA^T} = e^{-tA} J$$

$$\Leftrightarrow \quad J(I + tA^T + \tfrac{1}{2}t^2 A^{T^2} + \cdots) = (I - tA + \tfrac{1}{2}t^2 A^2) J.$$

$$\Leftrightarrow \quad J\left(\tfrac{1}{n!}(A^T)^n\right) = (-1)^n \tfrac{1}{n!} A^n \cdot J. \qquad (*).$$

In particular : $JA^T + AJ = 0.$

We prove it is equivalent to $(*)$:

$$J(A^T)^n = -AJ(A^T)^{n-1} = (-1)^1 A(J \cdot (A^T)^{n-1}) = \cdots = (-1)^n A^n \cdot J.$$

So

$$\mathfrak{g}_{Sp(2n)} = \mathfrak{sp}(2n) = \{A \in \mathfrak{gl}(2n, \mathbb{R}) \mid AJ + JA^T = 0\}$$
$$= \{A \in \mathfrak{gl}(2n, \mathbb{R}) \mid A^T J + JA = 0\}.$$

Ex 2.

1°. From Ado's theorem:

Every finite-dimensional real Lie algebra is isomorphic to a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. Every finite-dimensional complex Lie algebra is isomorphic to a complex subalgebra of $\mathfrak{gl}(n, \mathbb{C})$.

So we can assume $\mathfrak{g}$ is a matrix Lie algebra.

2°. We can choose a neighborhood U_0 of $0 \in \mathfrak{g}$ on which $\exp$ is a local diffeomorphism and $\exp^{-1}$ is well defined on $\exp(U_0)$. Let U be an open ball in $\mathfrak{g}$ s.t. $(\exp U)^2 (\exp U)^{-2} \subseteq \exp U_0$.

$\forall X, Y \in U$. $\gamma(t) := e^{tX} e^{tY}$. t can be choosed in a nghd of $[0,1]$. $\gamma(t) \in \exp(U)$.

$\exists\, Z(t) \in U_0$ s.t. $\exp(Z(t)) = \exp(tX)\exp(tY)$.

We have:

$$\frac{d}{dt}\exp(Z(t)) = X\exp(Z(t)) + \exp(Z(t))Y.$$

(a). If $X(t)$ is a smooth matrix-valued function, then

$$\frac{d}{dt}\exp(X(t)) = \exp(X(t))\left(\frac{I-\exp(-ad_{X(t)})}{ad_{X(t)}}\right)\frac{dX}{dt}$$

$$= \left(\frac{\exp(ad\, X(t)) - I}{ad(X(t))}\right)\frac{dX}{dt}\exp(X(t)).$$

From (a):

$$\left(\frac{\exp(ad_{Z(t)}) - I}{ad\, Z(t)}\right)\frac{dZ(t)}{dt}\exp(Z(t)) = X\exp(Z(t)) + \exp(Z(t))Y.$$

$Z(t) \in U_0$. $\exp$ is a local diffeomorphism near $Z(t)$.

$\frac{\exp(ad_{Z(t)}) - I}{ad\, Z(t)}$ is an invertible map on g.

$$\frac{dZ(t)}{dt} = \left(\frac{ad_{Z(t)}}{\exp(ad_{Z(t)}) - I}\right)\cdot\left(X + \exp(Z(t))\,Y\exp^{-1}(Z(t))\right).$$

$$= \left(\frac{ad_{Z(t)}}{\exp(ad_{Z(t)}) - I}\right)\cdot\left(X + Ad(e^{Z(t)})\,Y\right)$$

$$= \left(\frac{ad_{Z(t)}}{\exp(ad_{Z(t)}) - I}\right)\cdot\left(X + Ad(e^{tX})\,Ad(e^{tY})\,Y\right)$$

$$= \left(\frac{ad_{Z(t)}}{\exp(ad_{Z(t)}) - I}\right)\cdot\left(X + \exp(t\,ad(X))\exp(t\,ad(Y))\,Y\right)$$

$$= \left(\frac{ad_{Z(t)}}{\exp(ad_{Z(t)}) - I}\right)\cdot\left(X + \exp(t\,ad X)\,Y\right).$$

For $A = \mathrm{ad}\, Z(t)$.

$$A = \mathrm{Log}(I + (\exp(A) - I)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (e^{A} - I)^{n}$$

$$\Rightarrow \frac{\mathrm{ad}_{Z(t)}}{\exp(\mathrm{ad}_{Z(t)}) - I} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\exp(t\,\mathrm{ad}X)\exp(t\,\mathrm{ad}Y) - I)^{n-1}$$

$$\frac{dZ(t)}{dt} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\exp(t\,\mathrm{ad}X)\exp(t\,\mathrm{ad}Y) - I)^{n-1} (X + \exp(t\,\mathrm{ad}X)Y).$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \Big[\sum_{(i,j)\neq(0,0)}^{\infty} \frac{t^{i+j}}{i!\,j!} (\mathrm{ad}X)^{i} (\mathrm{ad}Y)^{j} \Big]^{n-1} (X + \sum_{i=0}^{\infty} \frac{t^{i}}{i!} (\mathrm{ad}X)^{i} Y).$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \Big[\sum_{i_k+j_k \geq 1} \frac{t^{i_1+j_1+\cdots+i_{n-1}+j_{n-1}}}{i_1!\,j_1!\cdots i_{n-1}!\,j_{n-1}!} [(X)^{i_1}, (Y)^{j_1}, \dots, (X)^{i_{n-1}}, (Y)^{j_{n-1}}, X]$$

$$+ \sum_{i_k+j_k \geq 1} \frac{t^{i_1+j_1+\cdots+i_{n-1}+j_{n-1}+i_n}}{i_1!\,j_1!\cdots i_{n-1}!\,j_{n-1}!\,i_n!} [(X)^{i_1}, (Y)^{j_1}, \dots, (X)^{i_{n-1}}, (Y)^{j_{n-1}}, (X)^{i_n}, Y]$$

using the notation that:

$$[X_n, \dots, X_3, X_2, X_1] = [X_n, \dots, [X_3, [X_2, X_1]\cdots]$$

$$[X_n^{(i_n)}, \dots, X_1^{(i_1)}] = [\underbrace{X_n, \dots X_n}_{i_n}, \dots, \underbrace{X_1, \dots X_1}_{i_1}]$$

Thus we have proved:

$$\exp(X)\exp(Y) = \exp(Z).$$

$$Z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{i_k+j_k \geq 1} \frac{1}{(i_1+j_1+\cdots+i_n+j_n)} \cdot \frac{1}{i_1!\,j_1!\cdots i_n!\,j_n!} [X^{i_1}, Y^{j_1}, \dots, X^{i_n}, Y^{j_n}]$$

In particular,

$$\exp(X)\exp(Y) = \exp(X + Y + \tfrac{1}{2}[X, Y] + \tfrac{1}{12}[X, [X, Y]] - \tfrac{1}{12}[Y, [X, Y]] + \cdots).$$

3°. Now for t small enough

$$\exp(-tX)\exp(-tY)\exp(tX)\exp(tY)$$

$$= \exp(-tX - tY + tX + tY + \tfrac{1}{2}t^2[X,Y] - \tfrac{t^2}{2}[X,Y] - \tfrac{t^2}{2}[Y,X]$$
$$+ \tfrac{1}{2}t^2[X,Y] + O(t^3)).$$

$$= \exp(t^2[X,Y] + O(t^3)).$$

4°. Proof of (a):

$$\frac{d}{dt}\exp(X(t)) = \exp(X(t))\left(\frac{I - \exp(-\mathrm{ad}_{X(t)})}{\mathrm{ad}_{X(t)}}\right)\frac{dX}{dt}$$

$$= \left(\frac{\exp(\mathrm{ad}_{X(t)}) - I}{\mathrm{ad}_{X(t)}}\right)\frac{dX}{dt}\exp(X(t)).$$

Define

$$\varphi(s,t) = \exp(-sX(t))\frac{\partial}{\partial t}\exp(sX(t))$$

$$\varphi(0,t) = 0. \qquad \varphi(1,t) = \int_0^1 \frac{\partial}{\partial s}\varphi(s,t)\,ds$$

$$\frac{\partial}{\partial s}\varphi(s,t)$$

$$= -X(t)\exp(-sX(t))\frac{\partial}{\partial t}\exp(sX(t))$$
$$+ \exp(-sX(t))\frac{\partial}{\partial t}[X(t)\exp(sX(t))]$$

$$= -\exp(-sX(t))X(t)\frac{\partial}{\partial t}\exp(sX(t)) + \exp(-sX(t))\frac{\partial}{\partial t}X(t)\exp(sX(t))$$
$$+ \exp(-sX(t))X(t)\frac{\partial}{\partial t}\exp(sX(t))$$

$$= \exp(-sX(t))\frac{\partial}{\partial t}X(t)\exp(sX(t))$$

$$= \mathrm{Ad}(\exp(-sX(t)))\frac{\partial}{\partial t}X(t) = \exp(-s\,\mathrm{ad}X(t))\frac{\partial}{\partial t}X(t).$$

$$\varphi(1,t) = \int_0^1 \exp(-s\,\mathrm{ad}\,X(t)) \frac{\partial}{\partial t} X(t)\, ds$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} (\mathrm{ad}\,X(t))^n \frac{\partial}{\partial t} X(t)\, ds$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n s^{n+1}}{(n+1)!} (\mathrm{ad}\,X(t))^n \frac{\partial}{\partial t} X(t) \Big|_{s=0}^{s=1}$$

$$= \left(\frac{I - \exp(-\mathrm{ad}\,X(t))}{\mathrm{ad}\,X(t)} \right) \frac{dX}{dt}$$

To show the second part, note that

$$L_{\exp(X(t))} = R_{\exp(X(t))} \circ \mathrm{Ad}(\exp(X(t)))$$
$$= R_{\exp(X(t))} \circ \exp(\mathrm{ad}\,X(t))$$

Another Proof for

$$\exp(tX)\exp(tY) = \exp(t(X+Y) + \tfrac{1}{2}t^2[X,Y] + O(t^3)).$$

Write $\exp(Z(t)) = \exp(tX)\exp(tY)$.

$$Z(0) = 0, \quad Z(t) = tZ_1 + t^2 Z_2 + O(t^3).$$

$\tilde{Z}_1, \tilde{Z}_2$ are the corresponding left-invariant vector fields.

We have Taylor's Theorem:

For $X \in \mathfrak{g}$, $\tilde{X}$ the corresponding left-invariant vector field, f a C^∞ function on G:

$$(\tilde{X}^n f)(g\exp(tX)) = \frac{d^n}{dt^n}(f(g\exp(tX))), \quad g \in G.$$

$$\Rightarrow f(\exp(Z(t)))$$

$$= \sum_{k=0}^{2} \frac{1}{k!} (t\tilde{Z}_1 + t^2 \tilde{Z}_2 + O(t^3))^k f(e) + O(t^3)$$

$$= f(e) + t(\tilde{Z}_1 f)(e) + t^2(\tfrac{1}{2}\tilde{Z}_1^2 + \tilde{Z}_2) f(e) + O(t^3).$$

$f(\exp(tX)\exp(sY))$

$= \sum_{k=0}^{2} \frac{1}{k!} s^k \tilde{Y}^k f(\exp(tX)) + O_t(s^3)$.

$= \sum_{k=0}^{2} \sum_{l=0}^{2} \frac{1}{k!} \frac{1}{l!} s^k t^l \tilde{X}^l \tilde{Y}^k f(e) + O_t(s^3) + O(t^3)$.

$f(\exp(Z(t)))$

$= f(\exp(tX)\exp(tY))$

$= f(e) + t(\tilde{X}+\tilde{Y})f(e) + t^2(\frac{1}{2}\tilde{X}^2 + \tilde{X}\tilde{Y} + \frac{1}{2}\tilde{Y}^2)f(e) + O(t^3)$.

This holds for f_g: $f_g(x) := f(gx)$.

$\Rightarrow \tilde{Z}_1 = \tilde{X}+\tilde{Y} \qquad \frac{1}{2}\tilde{Z}_1^2 + \tilde{Z}_2 = \frac{1}{2}\tilde{X}^2 + \tilde{X}\tilde{Y} + \frac{1}{2}\tilde{Y}^2$

$\Rightarrow Z_1 = X+Y. \qquad Z_2 = \frac{1}{2}[X, Y]$.

Ex 3.

(1).

(i). If $A \in M_n(\mathbb{R})$ is diagonalizable, that is.

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = A_0 \qquad P \in GL(n, \mathbb{R}).$$

$$\begin{aligned}\Rightarrow e^A &= Id + A + \frac{1}{2}A^2 + \cdots \\ &= Id + PA_0P^{-1} + \frac{1}{2}(PA_0P^{-1})^2 + \cdots \\ &= P(Id + A_0 + \frac{1}{2}A_0^2 + \cdots)P^{-1} \\ &= P e^{A_0} P^{-1}\end{aligned}$$

$$\det(e^A) = \det(e^{A_0}) = \prod_{i=1}^{n} e^{\lambda_i} = e^{\sum_{i=1}^{n}\lambda_i} = e^{tr(A_0)} = e^{tr(A)}$$

(ii). If $A \in M_n(\mathbb{R})$ is nilpotent, that is,

$$P^{-1}AP = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} = A_0.$$

$$e^{A_0} = Id + A_0 + \frac{1}{2}A_0^2 + \cdots = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$e^A = P e^{A_0} P^{-1}.$$

$$\det(e^A) = 1 = e^0 = e^{tr(A)}.$$

(iii). $\forall A \in M_n(\mathbb{R})$. $\exists N$ nilpotent & S diagonalizable

s.t. $A = N + S$ & $NS = SN$.

$$\Rightarrow \quad e^A = e^N e^S$$

$$\det e^A = \det(e^N)\det(e^S) = e^{tr(N)} \cdot e^{tr(S)}$$

$$= e^{tr(N+S)} = e^{tr(A)}.$$

(2). If $e^A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} =: B$

$$\Rightarrow \quad B = Id + A + \frac{1}{2}A^2 + \cdots$$

$$\Rightarrow \quad BA = AB. \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2a_{11} & -a_{12} \\ -2a_{21} & -a_{22} \end{pmatrix} = \begin{pmatrix} -2a_{11} & -2a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}. \quad \Rightarrow \quad a_{12} = a_{21} = 0$$

$$A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \quad \Rightarrow \quad e^A = \begin{pmatrix} e^{a_{11}} & 0 \\ 0 & e^{a_{22}} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow a_{11}, a_{22} \notin \mathbb{R}.$$

Ex 4.

$R(X,Y,Z,W) = g(R(X,Y)Z, W)$.

(i). $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$.

Let S be the operator:

$ST(X,Y,Z) := T(X,Y,Z) + T(Y,Z,X) + T(Z,X,Y)$.

$$\begin{aligned} SR(X,Y)Z &= S(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) \\ &= S(\nabla_X\nabla_Y Z) - S(\nabla_Y\nabla_X Z) - S(\nabla_{[X,Y]}Z). \\ &= S(\nabla_Z\nabla_X Y) - S(\nabla_Z\nabla_Y X) - S(\nabla_{[X,Y]}Z). \\ &= S(\nabla_Z[X,Y]) - S[\nabla_{[X,Y]}Z]. \\ &= S([Z,[X,Y]]) = 0. \end{aligned}$$

(ii). $R(X,Y)Z = -R(Y,X)Z$

$\Rightarrow R(X,Y,Z,W) = -R(Y,X,Z,W)$.

For the second part: Note that

$R(X,Y)f = X(Yf) - Y(Xf) - [X,Y]f = 0$.

$$\begin{aligned} \Rightarrow 0 &= R(X,Y)(\tfrac{1}{2}g(Z,Z)) \\ &= \tfrac{1}{2}X(Y\langle Z,Z\rangle) - \tfrac{1}{2}Y(X\langle Z,Z\rangle) - \tfrac{1}{2}[X,Y]\langle Z,Z\rangle \\ &= D_X\langle\nabla_Y Z, Z\rangle - D_Y\langle\nabla_X Z, Z\rangle - \langle\nabla_{[X,Y]}Z, Z\rangle \\ &= \langle\nabla_X\nabla_Y Z, Z\rangle - \langle\nabla_Y\nabla_X Z, Z\rangle - \langle\nabla_{[X,Y]}Z, Z\rangle \\ &\quad + \langle\nabla_Y Z, \nabla_X Z\rangle - \langle\nabla_X Z, \nabla_Y Z\rangle \\ &= \langle R(X,Y)Z, Z\rangle = R(X,Y,Z,Z). \end{aligned}$$

So $R(X,Y,Z+W,Z+W) = 0$

$\Rightarrow R(X,Y,Z,W) = -R(X,Y,W,Z)$.

(iii).

$$\begin{aligned} R(X,Y,Z,W) &= -R(Y,Z,X,W) - R(Z,X,Y,W). \\ &= R(Y,Z,W,X) + R(Z,X,W,Y) \\ &= -R(Z,W,Y,X) - R(W,Y,Z,X) - R(X,W,Z,Y) - R(W,Z,X,Y). \end{aligned}$$

$= 2R(W,Z,Y,X) + R(W,Y,X,Z) + R(X,W,Y,Z).$

$= 2R(W,Z,Y,X) - R(Y,X,W,Z).$

$\Rightarrow 2R(X,Y,Z,W) = 2R(Z,W,X,Y).$

$R(X,Y,Z,W) = R(Z,W,X,Y).$

Ex 5.

(i). If $\gamma(t)$ is a path from $e = I \in G$. $\gamma(0) = I$.

$$G \ni \gamma(t) = \begin{pmatrix} x(t) & y(t) \\ 0 & 1 \end{pmatrix}$$

$$\frac{d}{dt}\gamma(t) = \begin{pmatrix} x'(t) & y'(t) \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_G.$$

$$\forall \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_G \qquad \gamma(t) = \begin{pmatrix} e^{at} & bt \\ 0 & 1 \end{pmatrix} \subset G$$

for $t \in (-\varepsilon, \varepsilon)$.

$$\left.\frac{d}{dt}\right|_{t=0}\gamma(t) = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in T_eG.$$

$$\Rightarrow \mathfrak{g}_G = T_eG = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

(ii). For $p = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in G$. the corresponding left-invariant vector fields are

$$X_1(p) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \qquad X_2(p) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

$$= (x, 0) \qquad\qquad = (0, x).$$

$$= x\frac{\partial}{\partial x} \qquad\qquad = x\frac{\partial}{\partial y}$$

$g_{ij} = x^{-2}\delta_{ij}$. is the left invariant metric g on G:

$g = x^{-2}(dx \otimes dx + dy \otimes dy)$

(iii). Note that $[X_1, X_2] = X_2$ $[X_2, X_1] = -X_2$:

$\langle \nabla_{X_1} X_1, X_1 \rangle = \frac{1}{2} X_1 \langle X_1, X_1 \rangle = 0.$

$$\langle \nabla_{X_1} X_1, X_2 \rangle = \frac{1}{2} \{ X_1 \langle X_1, X_2 \rangle + X_1 \langle X_2, X_1 \rangle - X_2 \langle X_1, X_1 \rangle$$
$$- \langle X_1, [X_1, X_2] \rangle + \langle X_1, [X_2, X_1] \rangle$$
$$+ \langle X_2, [X_1, X_1] \rangle. \}.$$
$$= 0.$$

$\Rightarrow \nabla_{X_1} X_1 = 0.$

$\langle \nabla_{X_1} X_2, X_1 \rangle = X_1 \langle X_2, X_1 \rangle - \langle X_2, \nabla_{X_1} X_1 \rangle = 0$

$\langle \nabla_{X_1} X_2, X_2 \rangle = \frac{1}{2} X_1 \langle X_2, X_2 \rangle = 0.$

$\Rightarrow \nabla_{X_1} X_2 = 0.$

$\langle \nabla_{X_2} X_1, X_1 \rangle = \frac{1}{2} X_2 \langle X_1, X_1 \rangle = 0.$

$$\langle \nabla_{X_2} X_1, X_2 \rangle = \frac{1}{2} \{ - \langle X_2, [X_1, X_2] \rangle, + \langle X_1, [X_2, X_2] \rangle + \langle X_2, [X_2, X_1] \rangle \}$$
$$= \frac{1}{2} \cdot (-1-1) = -1$$

$\Rightarrow \nabla_{X_2} X_1 = -X_2.$

$\langle \nabla_{X_2} X_2, X_1 \rangle = X_2 \langle X_2, X_1 \rangle - \langle X_2, \nabla_{X_2} X_1 \rangle = 1$

$\langle \nabla_{X_2} X_2, X_2 \rangle = 0.$

$\Rightarrow \nabla_{X_2} X_2 = X_1.$

We use the Koszul formula. (Proved in HW2 EX9).

(iv). $K(p)(X_1, X_2) = \langle R(X_1, X_2) X_2, X_1 \rangle$
$$= \langle \nabla_{X_1} \nabla_{X_2} X_2 - \nabla_{X_2} \nabla_{X_1} X_2 - \nabla_{[X_1, X_2]} X_2, X_1 \rangle$$
$$= -1$$

$\Rightarrow K(p) \equiv -1.$