2024秋微分流形第五次作业解答

EXERCISE 1. (WHITNEY'S APPROXIMATION THEOREM)

Pf. Choose one cordinate chart covering $\{(\varphi_\alpha, U_\alpha)\}$ s.t.

$\overline{U_\alpha} \overset{cpt}{\subset} M$, $\forall \alpha$. $\varepsilon: M \to \mathbb{R}$ cts, so: $\varepsilon_\alpha := \inf_{U_\alpha} \varepsilon > 0$.

Step 1: $\forall f \in C(M)$, $\exists g \in C^\infty(M)$ s.t. $|f - g| < \frac{\varepsilon}{\sqrt{n}}$.

Take P.O.U. $\{\rho_\alpha\}$ subject to $\{U_\alpha\}$. Let $f_\alpha :=$ $f \circ \varphi_\alpha^{-1} \in C(U_\alpha)$ & $\sup_{U_\alpha} |f_\alpha| < +\infty$. $\varphi_\alpha(U_\alpha)$ cpt.

$\Rightarrow$ Standard modifying steps in REAL ANALYSIS:

$\exists f_\alpha^{\varepsilon_\alpha} \in C^\infty(U_\alpha)$ s.t. $|f_\alpha^{\varepsilon_\alpha} - f_\alpha| < \frac{\varepsilon_\alpha}{\sqrt{n}} \leq \frac{\varepsilon \circ \varphi_\alpha^{-1}}{\sqrt{n}}$.

Take $g = \sum_\alpha \rho_\alpha \cdot (f_\alpha^{\varepsilon_\alpha} \circ \varphi_\alpha)$, $g \in C^\infty(M)$ (due to P.O.U. locally finite), and:

$$
\begin{aligned}
|f - g| &= \left| \sum_\alpha \rho_\alpha (f - f_\alpha^{\varepsilon_\alpha} \circ \varphi_\alpha) \right| \\
&\leq \sum_\alpha \rho_\alpha \left| (f_\alpha - f_\alpha^{\varepsilon_\alpha}) \circ \varphi_\alpha \right| \\
&\leq \sum_\alpha \rho_\alpha \frac{\varepsilon \circ \varphi_\alpha^{-1} \circ \varphi_\alpha}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_\alpha \rho_\alpha \cdot \varepsilon = \frac{\varepsilon}{\sqrt{n}}.
\end{aligned}
$$

Step 2: $\forall F \in C(M, \mathbb{R}^k)$, $\exists G \in C^\infty(M, \mathbb{R}^k)$ s.t. $\|F - G\| < \varepsilon$.

Write $F=(F^1,\cdots,F^k)$, $F^i \in C(M)$, $\forall 1\leq i\leq k$.

From step 1, $\exists G^i \in C^\infty(M)$, s.t. $|F^i-G^i|<\frac{\varepsilon}{\sqrt{n}}$.

Take $G=(G^1,\cdots,G^k)\in C^\infty(M,\mathbb{R}^k)$, so:

$$\|F-G\| = \left(\sum_{i=1}^{n}|F^i-G^i|^2\right)^{1/2}$$
$$< \left(\sum_{i=1}^{n}\frac{\varepsilon^2}{n}\right)^{1/2} = (\varepsilon^2)^{1/2} = \varepsilon. \quad \#$$

<u>Rmk</u>. One can choose $F=G$ on a closed subset of M.
(See Prof. Zuoqing Wang's Lecture in 2023)

EXERCISE 2.

<u>Sol</u> Let $i_1: \mathbb{S}^2 \hookrightarrow \mathbb{D}^3$, $i_2: \mathbb{D}^3 \hookrightarrow \mathbb{R}^3$. $\mathbb{D}^3$: closed unit ball in $\mathbb{R}^3$
& $\partial\mathbb{D}^3 = \mathbb{S}^2$. So: $i = i_2\circ i_1: \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$, and:

$$\int_{\mathbb{S}^2} i^*\theta = \int_{\mathbb{S}^2} i_1^* i_2^* \theta$$
$$\overset{\text{Stokes}}{=\!=} \int_{\mathbb{D}^3} d(i_2^*\theta) = \int_{\mathbb{D}^3} i_2^*(d\theta)$$

Symmetry of $\mathbb{D}^3$:

$\int_{\mathbb{D}^3} x\,dx\,dy\,dz = 0$

$$= \int_{\mathbb{D}^3} (2x+2)\,dx\wedge dy\wedge dz$$
$$= \int_{\mathbb{D}^3} 2\,dx\wedge dy\wedge dz$$
$$= 2\cdot \mathrm{volume}(\mathbb{D}^3) = \frac{8}{3}\pi.$$

EXERCISE 3.

Pf. (1) Prove by induction. Suppose for some $k \in \mathbb{N}^*$,

$$\alpha \wedge (d\alpha)^k - \beta \wedge (d\beta)^k$$
$$= (\alpha - \beta) \wedge \sum_{j=0}^{k} (d\alpha)^j \wedge (d\beta)^{k-j}$$
$$+ d\left(\alpha \wedge \beta \wedge \sum_{j=0}^{k-1} (d\alpha)^j \wedge (d\beta)^{k-j-1}\right),$$

then for $k+1$:

$\alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha$
$\alpha \in \Omega^r$
$\beta \in \Omega^s$

$$\alpha \wedge (d\alpha)^{k+1} - \beta \wedge (d\beta)^{k+1}$$
$$= [\alpha \wedge (d\alpha)^k - \beta \wedge (d\beta)^k] \wedge d\alpha$$
$$+ \beta \wedge (d\beta)^k \wedge (d\alpha - d\beta)$$

$d(\alpha \wedge \beta)$
$= d\alpha \wedge \beta$
$+ (-1)^r \alpha \wedge d\beta$
$\alpha \in \Omega^r$
$\beta \in \Omega^s$

$$= (\alpha - \beta) \wedge \sum_{j=0}^{k} (d\alpha)^{j+1} \wedge (d\beta)^{k-j}$$
$$+ d\left(\alpha \wedge \beta \wedge \sum_{j=1}^{k-1} (d\alpha)^j \wedge (d\beta)^{k-j-1}\right) \wedge d\alpha$$
$$+ (d\alpha - d\beta) \wedge \beta \wedge (d\beta)^k$$

$(\alpha - \beta) \wedge \beta = \alpha \wedge \beta$

$d(d\alpha)^k = 0$

$$= \underline{(\alpha - \beta) \wedge \sum_{j=1}^{k+1} (d\alpha)^j \wedge (d\beta)^{k+1-j}}$$
$$+ d\left(\alpha \wedge \beta \wedge \sum_{j=1}^{k-1} (d\alpha)^{j+1} \wedge (d\beta)^{k-j-1}\right)$$
$$+ d\left(\underline{(\alpha - \beta) \wedge \beta \wedge (d\beta)^k}\right) + \underline{(\alpha - \beta) \wedge (d\beta)^{k+1}}$$
$$= (\alpha - \beta) \wedge \sum_{j=0}^{k+1} (d\alpha)^j \wedge (d\beta)^{k+1-j}$$
$$+ d\left(\alpha \wedge \beta \wedge \sum_{j=0}^{k} (d\alpha)^j \wedge (d\beta)^{k-j}\right)$$

still holds, only to check $n=2$:

$$\alpha\wedge(d\alpha)^2-\beta\wedge(d\beta)^2$$
$$=(\alpha-\beta)\wedge(d\alpha)^2+(\alpha-\beta)\wedge d\alpha\wedge d\beta$$
$$+(\alpha-\beta)\wedge(d\beta)^2$$
$$+\beta\wedge(d\alpha)^2-(\alpha-\beta)\wedge d\alpha\wedge d\beta-\alpha\wedge(d\beta)^2$$
$$=(\alpha-\beta)\wedge\sum_{j=0}^{2}(d\alpha)^j\wedge(d\beta)^{2-j}$$
$$+d\alpha\wedge\beta\wedge d\alpha-\alpha\wedge d\beta\wedge d\alpha+\underbrace{\alpha\wedge\beta\wedge d(d\alpha)}_{=0}$$
$$+d\alpha\wedge\beta\wedge d\beta-\alpha\wedge(d\beta)^2+\underbrace{\alpha\wedge\beta\wedge d(d\beta)}_{=0}$$
$$=(\alpha-\beta)\wedge\sum_{j=0}^{2}(d\alpha)^j\wedge(d\beta)^{2-j}$$
$$+d(\alpha\wedge\beta\wedge d\alpha)+d(\alpha\wedge\beta\wedge d\beta)$$

also holds. #

(2) First prove: for any $\omega\in\Omega^{2n+1}(M)$, $\omega=\alpha\wedge\iota_X\omega$.

Pf Using HW3Ex8, $\exists$ local chart s.t. $X=\partial_1$

So $\alpha=dx^1+\sum_{j=2}^{2n+1}\alpha_j dx^j$.

Write $\omega=\lambda dx^1\wedge\cdots\wedge dx^{2n+1}$, so:

$$\iota_X\omega=\lambda\sum_{j=1}^{2n+1}(-1)^{j-1}dx^j(X)dx^1\wedge\cdots\wedge\widehat{dx^j}\wedge\cdots\wedge dx^{2n+1}$$
$$=\lambda dx^2\wedge\cdots\wedge dx^{2n+1},$$

then: $\alpha \wedge \iota_X \omega = (dx^1 + \sum_{j=2}^{2n+1} \alpha_j dx^j) \wedge (\lambda dx^2 \wedge \cdots \wedge dx^{2n+1})$

$= \lambda dx^1 \wedge \cdots \wedge dx^{2n+1} = \omega .$ #

So: By Stoke's formula,

$$\int_M \alpha \wedge (d\alpha)^n = \int_M \beta \wedge (d\beta)^n + \int_M (\alpha - \beta) \wedge \sum_{j=0}^{n} (d\alpha)^j \wedge (d\beta)^{n-j}$$

Only to show: $(\alpha - \beta) \wedge \sum_{j=0}^{n} (d\alpha)^j \wedge (d\beta)^{n-j} = 0$

$\Longleftrightarrow \iota_X\left((\alpha-\beta)\wedge \sum_{j=0}^{n} (d\alpha)^j \wedge (d\beta)^{n-j}\right) = 0$

$$\left.\begin{array}{l} \iota_X(\alpha - \beta) = \alpha(X) - \beta(X) = 1 - 1 = 0; \\ \mathcal{L}_X \alpha = \iota_X d\alpha + d(\iota_X \alpha) = \iota_X d\alpha = 0; \\ \mathcal{L}_X \beta = \iota_X d\beta + d(\iota_X \beta) = \iota_X d\beta = 0 \end{array}\right\}$$

$\Rightarrow \iota_X\left((\alpha-\beta)\wedge \sum_{j=0}^{n} (d\alpha)^j \wedge (d\beta)^{n-j}\right)$

$= \underbrace{\iota_X(\alpha-\beta)}_{=0} \wedge \sum_{j=0}^{n} (d\alpha)^j \wedge (d\beta)^{n-j}$

$- (\alpha-\beta) \wedge \iota_X\left(\sum_{j=0}^{n} (d\alpha)^j \wedge (d\beta)^{n-j}\right)$

$= -(\alpha-\beta) \wedge \Big(\sum_{j=0}^{n} \sum_{k=0}^{j-1} (d\alpha)^k \wedge \iota_X d\alpha \wedge (d\alpha)^{j-k-1} \wedge (d\beta)^{n-j}$

$+ \sum_{j=0}^{n} \sum_{k=0}^{j-1} (d\alpha)^{n-j} \wedge (d\beta)^k \wedge \iota_X d\beta \wedge (d\beta)^{j-k-1} \Big)$

$= 0$ #

<u>Rmk</u> $\alpha(X)=1$ 实际上说明 TM 有处处非零的截面.

conclusion 1: $\chi(M)=0$

conclusion 2: de Rham 定理

局部可以变为乘积流形 (分裂)

(诱导同调群的分裂)

$\omega=\alpha\wedge \iota_X\omega \Rightarrow \Omega^n(M)\simeq\Omega^{n-1}(M).$

EXERCISE 4.

Pf. (1) $\forall x\in M$, $\omega_x := \omega|_{T_xM} : T_xM\times T_xM \to \mathbb{R}$ bilinear & anti-symmetric. Choose one basis of T_xM as $\{u_i, v_i\}_{i=1}^n$, the related dual basis $\{u^i, v^i\}_{i=1}^n$ s.t.: (Standard Linear Algebra Conclusion)

$$\omega_x = \sum_{i=1}^n b_i (u^i\otimes v^i - v^i\otimes u^i) = \sum_{i=1}^n b_i u^i\wedge v^i,$$

where $b_i\in\mathbb{R}$. So:

$$\omega^n|_{T_xM} = (\omega_x)^n = \overbrace{\omega_x\wedge\cdots\wedge\omega_x}^{n}$$

$$= \left(\sum_{i=1}^n b_i u^i\wedge v^i\right)\wedge\cdots\wedge\left(\sum_{i=1}^n b_i u^i\wedge v^i\right)$$

$$= n!\left(\prod_{i=1}^n b_i\right)\underbrace{u^1\wedge v^1\wedge\cdots\wedge u^n\wedge v^n}_{\substack{\text{standard volume form}\\ \text{w.r.t. basis } \{u_i, v_i\}_{i=1}^n}}$$

So: ω non-degenerate $\Leftrightarrow \forall x$, $\omega_x = \sum_{i=1}^n b_i u^i\wedge v^i$ with $b_i\neq 0$, $\forall 1\le i\le n$

$\Leftrightarrow \prod_{i=1}^n b_i \neq 0 \Leftrightarrow \forall x, (\omega_x)^n$ is not-vanishing

$\Leftrightarrow \omega^n$ is nowhere vanishing.

(2) $$\begin{aligned}\{F,G\}\,\omega^n &= \omega(X_F, X_G)\,\omega^n \\ &= -dF(X_G)\,\omega^n \\ &= -\iota_{X_G}(\underbrace{dF\wedge\omega^n}_{=0}) - dF\wedge\iota_{X_G}\omega^n \\ &= -dF\wedge\Big(\sum_{j=0}^{n-1}\omega^j\wedge\underbrace{\iota_{X_G}\omega}_{=-dG}\wedge\omega^{n-1-j}\Big) \\ &= dF\wedge\Big(\sum_{j=0}^{n-1}\omega^j\wedge dG\wedge\omega^{n-1-j}\Big) \\ &= dF\wedge\Big(\sum_{j=0}^{n-1} dG\wedge\omega^j\wedge\omega^{n-1-j}\Big) \\ &= n\,dF\wedge dG\wedge\omega^{n-1} \\ &= n\,d(F\wedge dG\wedge\omega^{n-1})\end{aligned}$$

$$\iota_X(\alpha\wedge\beta) = (\iota_X\alpha)\wedge\beta + (-1)^r\alpha\wedge(\iota_X\beta)$$
$\alpha\in\Omega^r$
$\beta\in\Omega^s$
$X\in\Gamma^\infty(TM)$

So $\int_M \{F,G\}\,\omega^n = 0$. #

EXERCISE 5. (Generalized FUBINI'S THEOREM on Smooth Manifolds)

Pf. Due to the structure of product manifolds. let $\{U_\alpha\}, \{V_\gamma\}$ be the local chart coverings of M, N, and $\{\rho_\alpha\}, \{\varphi_\gamma\}$ be P.O.Us subject to $\{U_\alpha\}, \{V_\gamma\}$, then:

- $\{U_\alpha \times V_\gamma\}$ is a local chart covering of $M \times N$;
- $\{(\rho_\alpha \circ \pi_M)\cdot(\varphi_\gamma \circ \pi_N)\}$ is a P.O.U. belonging to $\{U_\alpha \times V_\gamma\}$.

So:
$$\int_{M\times N} \alpha \times \beta = \sum_{\alpha,\gamma} \int_{M\times N} (\rho_\alpha \circ \pi_M)\cdot(\varphi_\gamma \circ \pi_N)\, \pi_M^* \alpha \wedge \pi_N^* \beta$$
$$= \sum_{\alpha,\gamma} \int_{U_\alpha \times V_\gamma} (\rho_\alpha \circ \pi_M)\cdot(\varphi_\gamma \circ \pi_N)\, \pi_M^* \alpha \wedge \pi_N^* \beta$$

$f_\alpha \circ \pi^M_{\alpha,\gamma} \circ i_\alpha = Id$

$g_\gamma \circ \pi^N_{\alpha,\gamma} \circ i_\gamma = Id$

$i_\alpha : \tilde{U}_\alpha \to \tilde{U}_\alpha \times \tilde{V}_\gamma$

$i_\gamma : \tilde{V}_\gamma \to \tilde{U}_\alpha \times \tilde{V}_\gamma$

$\Downarrow$

$i_\alpha^* (\pi^M_{\alpha,\gamma})^* = (f_\alpha^{-1})^*$

$i_\gamma^* (\pi^N_{\alpha,\gamma})^* = (g_\gamma^{-1})^*$

let: $\begin{cases} f_\alpha : U_\alpha \to \tilde{U}_\alpha \subset \mathbb{R}^m \\ g_\gamma : V_\gamma \to \tilde{V}_\gamma \subset \mathbb{R}^n \end{cases}$ cordinate maps (diffeomorphisms)

& $\begin{cases} \pi^M_{\alpha,\gamma} : \tilde{U}_\alpha \times \tilde{V}_\gamma \to U_\alpha, & \pi^M_{\alpha,\gamma} = \pi_M \circ (f_\alpha \times g_\gamma)^{-1} \\ \pi^N_{\alpha,\gamma} : \tilde{U}_\alpha \times \tilde{V}_\gamma \to V_\gamma, & \pi^N_{\alpha,\gamma} = \pi_N \circ (f_\alpha \times g_\gamma)^{-1} \end{cases}$

$$\overset{def}{=} \sum_{\alpha,\gamma} \int_{\tilde{U}_\alpha \times \tilde{V}_\gamma} (\rho_\alpha \circ \pi^M_{\alpha,\gamma})(\varphi_\gamma \circ \pi^N_{\alpha,\gamma}) \underbrace{(\pi^M_{\alpha,\gamma})^* \alpha \wedge (\pi^N_{\alpha,\gamma})^* \beta}$$

实际上 $= i_\alpha^* (\pi^M_{\alpha,\gamma})^* \alpha \wedge i_\gamma^* (\pi^M_{\alpha,\gamma})^* \beta = (f_\alpha^{-1})^* \alpha \wedge (g_\gamma^{-1})^* \beta$

$$\overset{Fubini}{=\!=} \sum \left(\int_{\tilde{U}_\alpha} (\rho_\alpha \circ f_\alpha^{-1})(f_\alpha^{-1})^* \alpha \right)\left(\int_{\tilde{V}_\gamma} (\varphi_\gamma \circ g_\gamma^{-1})(g_\gamma^{-1})^* \beta \right)$$

$$= \left(\sum_\alpha \int_{\tilde{U}_\alpha} (\rho_\alpha \circ f_\alpha^{-1})(f_\alpha^{-1})^* \alpha \right)\left(\sum_\gamma \int_{\tilde{V}_\gamma} (\varphi_\gamma \circ g_\gamma^{-1})(g_\gamma^{-1})^* \beta \right)$$

$$\overset{def}{=} \left(\int_M \alpha \right)\left(\int_N \beta \right) \quad \#$$