Ex 1: Cayley transform.

Write B^2 & $\mathbb{H}^2$ as subsets of $\mathbb{C}$: $(x, y) \mapsto x+iy$.

Consider $F: \ z \mapsto i\dfrac{1+z}{1-z}$

For $|z|<1$. $z = x+iy$. $x^2+y^2<1$.

$$F(z) = i\,\frac{1+x+iy}{1-x-iy} = \frac{-2y + (1-x^2-y^2)i}{(1-x)^2+y^2} \in \mathbb{H}^2.$$

Thus we define:

$$F: \mathbb{B}^2 \to \mathbb{H}^2. \qquad (x,y) \mapsto \left(\frac{-2y}{(1-x)^2+y^2},\ \frac{1-x^2-y^2}{(1-x)^2+y^2}\right).$$

F^{-1} is defined by $F^{-1}(z) = \dfrac{z-i}{z+i}$: $\mathbb{H}^2 \to \mathbb{B}^2$.

$$F^{-1}(z) = \frac{x+i(y-1)}{x+i(y+1)} \qquad \frac{|x+i(y-1)|^2}{|x+i(y+1)|^2} = \frac{x^2+(y-1)^2}{x^2+(y+1)^2} < 1 \text{ if } y>0.$$

$F(z) = i\dfrac{1+z}{1-z} = 0 \Rightarrow z=-1 \notin B^2. \Rightarrow F$ is injective on B^2.

$\forall w \in \mathbb{H}^2$. $z = \dfrac{w-i}{w+i} \in B^2$ s.t. $F(z) = i\dfrac{1+\frac{w-i}{w+i}}{1-\frac{w-i}{w+i}} = w \Rightarrow F$ is surjective.

For F, with no singularities in B^2. it is indeed smooth.

Similar case for F^{-1}.

Now F is a smooth diffeomorphism : $B^2 \to \mathbb{H}^2$.

$$\frac{\partial F^1}{\partial x} = \frac{-2y\cdot 2(1-x)}{((1-x)^2+y^2)^2} \qquad \frac{\partial F^1}{\partial y} = \frac{-2((1-x)^2+y^2)+4y^2}{((1-x)^2+y^2)^2} = \frac{-2((1-x)^2-y^2)}{((1-x)^2+y^2)^2}.$$

$$\frac{\partial F^2}{\partial x} = \frac{-2x((1-x)^2+y^2)+2(1-x)(1-x^2-y^2)}{((1-x)^2+y^2)^2} = \frac{2(1-x)^2-2y^2}{((1-x)^2+y^2)^2}$$

$$\frac{\partial F^2}{\partial y} = \frac{-2y((1-x)^2+y^2)-2y(1-x^2-y^2)}{((1-x)^2+y^2)^2} = \frac{-4y(1-x)}{((1-x)^2+y^2)^2}.$$

$$\Rightarrow g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \frac{4}{(1-(x^2+y^2))^2}$$

$$g'(F_*(\tfrac{\partial}{\partial x}), F_*\tfrac{\partial}{\partial x}) = \left(\frac{(1-x)^2+y^2}{1-(x^2+y^2)^2}\right)^2 \cdot \left[\frac{16y^2(1-x)^2}{((1-x)^2+y^2)^4} + \frac{4\cdot((1-x)^2-y^2)^2}{((1-x)^2+y^2)^4} \right]$$

$$= \frac{((1-x)^2+y^2)^2}{(1-(x^2+y^2))^2} \cdot \frac{4\cdot((1-x)^2+y^2)^2}{((1-x)^2+y^2)^4}$$

$$= \frac{4}{(1-(x^2+y^2)^2)^2}$$

$$= g(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}).$$

$$g'(F_*(\tfrac{\partial}{\partial x}), F_*(\tfrac{\partial}{\partial y})) = \left(\frac{(1-x)^2+y^2}{1-(x^2+y^2)^2}\right)^2 \cdot \left[\frac{-2y\cdot 2(1-x)}{((1-x)^2+y^2)^2} \frac{-2((1-x)^2-y^2)}{((1-x)^2+y^2)^2} + \frac{2(1-x)^2-2y^2}{((1-x)^2+y^2)^2} \frac{-4y(1-x)}{((1-x)^2+y^2)^2} \right]$$

$$= 0 = g(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}).$$

$$g'(F_*(\tfrac{\partial}{\partial y}), F_*(\tfrac{\partial}{\partial y})) = \left(\frac{(1-x)^2+y^2}{1-(x^2+y^2)^2}\right)^2 \cdot \left[\frac{4\cdot((1-x)^2-y^2)^2}{((1-x)^2+y^2)^4} + \frac{16y^2(1-x)^2}{((1-x)^2+y^2)^4} \right]$$

$$= \frac{4}{(1-(x^2+y^2)^2)^2}$$

$$= g(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial y}).$$

Thus F preserves the metrics.

EX2.

(1). $q := (0,1) \in \mathbb{R}^2$. If $p = (x, y, s, t) \in \mathbb{R}^4$ s.t. $\Phi(p) = q$.

$x^2+y = 0$. $y^2+s^2+t^2 = 1$.

If $(0,1) \in \mathbb{R}^2$ is not a regular value of Φ. $\exists\, p \in \Phi^{-1}(q)$ s.t.

$$d\Phi_p = \begin{pmatrix} 2x & 1 & 0 & 0 \\ 2x & 2y+1 & 2s & 2t \end{pmatrix}$$ not surjective.

$\Rightarrow$ rank$(d\Phi_p) < 2$. $2x(2y+1) - 2x = 4xs = 4xt = 2s = 2t = 0$

$y^2+s^2+t^2 = 1 \Rightarrow y^2 = 1$. $x^2+y = 0 \Rightarrow x^2 = 1 = -y$.

which is a contradiction to $2x(2y+1)-2x = 4xy = 0$.
$\Rightarrow \forall p \in \Phi^{-1}(q)$. p is a regular point. $q=(0,1)$ is a regular value.

(2). $\Phi^{-1}(0,1)$ is a smooth submanifold of $\mathbb{R}^4$ of dim 2.
$\exists$ smooth immersion $\iota: \Phi^{-1}(0,1) \hookrightarrow \mathbb{R}^4$. Set smooth projection

$$p: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$
$$(x,y,s,t) \longmapsto (x,s,t).$$

Then $p\circ\iota: \Phi^{-1}(0,1) \to S \subset \mathbb{R}^3$. $S = \{(x,s,t) \mid x^4+s^2+t^2=1\}$.
1 is regular value of $x^4+s^2+t^2$. $\Rightarrow S$ is smooth submanifold of $\mathbb{R}^3$. $p\circ\iota$ is diffeomorphism since

$$\psi: S \longrightarrow \Phi^{-1}(0,1).$$
$$(x,s,t) \longmapsto (x,-x^2,s,t).$$

is also smooth & the inverse map.
Now only need to construct the diffeomorphism from S to S^2

$$\phi: S=\{x^4+s^2+t^2=1\} \to S^2 := \{a^2+b^2+c^2=1\}.$$
$$(x,s,t) \longmapsto \left(x, \frac{s}{\sqrt{1+x^2}}, \frac{t}{\sqrt{1+x^2}}\right).$$
$$\phi^{-1}: S^2 \longrightarrow S$$
$$(a,b,c) \longmapsto (a, \sqrt{1+a^2}\,b, \sqrt{1+a^2}\,c).$$

These maps are smooth.

Ex 3.

If F is smooth submersion : $N \to \mathbb{R}^k$. F is open map
$\Rightarrow F(N)$ open. & N compact + F continuous $\Rightarrow F(N)$ compact.
$F(N) \subset \mathbb{R}^k \Rightarrow F(N)$ is closed. $F(N) \neq \phi \Rightarrow F(N) = \mathbb{R}^k$.
This gives the contradiction.

Ex 4.

(1). Connectedness :

$\forall p.q \in \mathbb{R}^m \setminus N$. if $\tilde{C}_{p.q}(t)$ is a path from p to q. $t \in [0.1]$.
By Whitney Approximation thm, there is a smooth curve $C'_{p.q}(t)$ homotopic to $\tilde{C}_{p.q}(t)$.
From Transversality Homotopy thm, $\exists$ $C_{p.q}(t)$ smooth & is transverse to N & homotopic to $C'_{p.q}(t)$.

$C_{p.q}(0) = p$. $C_{p.q}(1) = q$. $\dim N + \dim C_{p.q} \leq m-2 < m$.

$\Rightarrow$ $N \cap C_{p.q}([0.1]) = \phi$. $C_{p.q} \subset \mathbb{R}^m \setminus N$.

(2). Simply connectedness.

If $C_1(t)$. $C_2(t)$ are two loops $\subset \mathbb{R}^m \setminus N$, $\exists$ $h(s.t) = h_s(t)$ homotopy s.t. $h_0(t) = C_1$. $h_1 = C_2$. ($\mathbb{R}^m$ is simply connected).
With similar discussion, we can perturb h to make it smooth & intersect N transversally.

$$\dim N + 2 \leq m-1 < m = \dim \mathbb{R}^m.$$

$\Rightarrow$ Surface $h \cap N = \phi$. $\Rightarrow$ $\mathbb{R}^m \setminus N$ is simply connected.

EX 5.

(1). The fixed points of r_θ are $N = (0.0,1)$. $S = (0.0.-1)$:

$$\begin{cases} x\cos\theta - y\sin\theta = x. \\ x\sin\theta + y\cos\theta = y \end{cases} \Rightarrow (x.y) = (0.0).$$

At $(0.0.1) = N$. consider stereographic projection:

$\sigma_2 : S^2 \setminus S \to \mathbb{R}^2$. $(x.y.z) \mapsto \left(\frac{x}{1+z}, \frac{x}{1+z}\right)$.

$\sigma_2 \circ r_\theta \circ \sigma_2^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$.

$r_\theta \circ \sigma_2^{-1}((u.v)) = r_\theta\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1} \right)$

$\sigma_2 \circ r_\theta \circ \sigma_2^{-1}((u.v)) = (u\cos\theta - v\sin\theta, u\sin\theta + v\cos\theta).$

$\Rightarrow d\gamma_\theta(N) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

If 1 is the eigenvalue: $(\cos\theta - 1)^2 + \sin^2\theta = 0$ contradiction to $\theta \neq 2k\pi$.

At $S = (0.0.-1)$. $\sigma_1 : S^2 \setminus \{N\} \to \mathbb{R}^2$. $\sigma_1(x.y.z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$.

$\sigma_1^{-1}(u.v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}. \frac{u^2+v^2-1}{u^2+v^2+1}\right)$.

$\sigma_1 \circ \gamma_\theta \circ \sigma_1^{-1}((u.v)) = (u\cos\theta - v\sin\theta, u\sin\theta + v\cos\theta)$. Similar case !

In sum, N. S are non-degenerate.

(2).

1°. As subspaces of $V \times V$, Γ_F intersects Δ transversally iff $\Gamma_F + \Delta = V \times V$.

From linear algebra,

$$\dim(\Gamma_F + \Delta) = \dim(\Gamma_F) + \dim(\Delta) - \dim(\Gamma_F \cap \Delta).$$

So $\Gamma_F + \Delta = V \times V$ iff $\dim(\Delta \cap \Gamma_F) = 0$.

$\Leftrightarrow \Delta \cap \Gamma_F = \{(0.0)\}$.

$\Leftrightarrow F(v) = v$ iff $v = 0$. $\Leftrightarrow$ 1 is not an eigenvalue of F.

2°. If p is a fixed point of F, consider the local coordinate $(\varphi. U. V)$. $p \in U \underset{\text{open}}{\subseteq} M$. V open in $\mathbb{R}^m$.

WLOG, set V as an open ball B centered at 0 and $\varphi(p) = 0$.

Then $f = \varphi \circ F \circ \varphi^{-1} : B \to B$.

Define $g = f - Id : B \to B$. $g(0) = \varphi(p) - 0 = 0$.

Since F is Lefschetz. $\det(dg(0)) \neq 0$. g is a local diffeomorphism near 0. $\exists$ small neighbourhood of 0, U_1 s.t. g maps U_1 diffeomorphically to $g(U_1)$.

That gives the fact that p is the only fixed point

in $\varphi^{-1}(U_1)$. Indeed if $\exists q \in \varphi^{-1}(U_1)$. $F(q) = q$. $q \neq p$.
$g(\varphi(q)) = \varphi(q) - \varphi(q) = 0$. $\lightning$.

So far, we have proved that all fixed points of F are isolated. The compactness of M completes the proof.

(3). With the computation in (1).

$$\det\left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} - Id_{2\times 2}\right) = (\cos\theta - 1)^2 + \sin^2\theta > 0. \qquad \theta \neq 2k\pi.$$

$\Rightarrow L(\gamma_\theta) = 2.$

EX6.

(1). Consider the map $F: M_{2n\times 2n}(\mathbb{R}) \longrightarrow H = \{H \in M_{2n\times 2n}(\mathbb{R}) : H + H^T = 0\}$.

$F(A) = AJ_0A^T - J_0$. Then $Sp(2n) = F^{-1}(0)$.

$$\begin{aligned} dF_A(B) &= \frac{d}{dt}\Big|_{t=0} F(A+tB) \\ &= \frac{d}{dt}\Big|_{t=0} (A+tB)J_0(A+tB)^T = BJ_0A^T + AJ_0B^T \\ &= BJ_0A^T - (BJ_0A^T)^T \in H \end{aligned}$$

$AJ_0A^T = J_0$ is invertible $\Rightarrow J_0A^T$ is invertible.

$dF_A : M_{2n\times 2n}(\mathbb{R}) \longrightarrow H$ is surjective $\forall A \in F^{-1}(0)$.

$\Rightarrow F^{-1}(0) = Sp(2n)$ is a submanifold.

(2). $\forall A \in Sp(2n)$. $T_A Sp(2n) = Ker(dF_A)$

$$\begin{aligned} \dim(Sp(2n)) &= \dim(M_{2n\times 2n}(\mathbb{R})) - \dim H \\ &= 4n^2 - \frac{2n(2n-1)}{2} = n(2n+1). \end{aligned}$$

EX7.

$\forall (p,q) \in N_1 \times N_2$. $p \in N_1$. $q \in N_2$. $\exists$ local chart (φ_1, U_1, V_1) near p.
(φ_2, U_2, V_2) near q s.t. $\varphi_1(U_1 \cap N_1) = \{x \in V_1 \subset \mathbb{R}^{m_1} \mid x_{n_1+1} = \cdots = x_{m_1} = 0\}$.
$\varphi_2(U_2 \cap N_2) = \{x \in V_2 \subset \mathbb{R}^{m_2} \mid x_{n_2+1} = \cdots = x_{m_2} = 0\}$.

Take $U_1 \times U_2$ as a neighbourhood of $(p.q)$ in $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$.

$$\varphi: U_1 \times U_2 \to V_1 \times V_2 \overset{open}{\subset} \mathbb{R}^{m_1+m_2}$$

$$(y_1 \times y_2) \longmapsto (\varphi_1(y_1). \varphi_2(y_2)).$$

$\varphi(p.q) = (\varphi_1(p). \varphi_2(q))$. & φ is homeomorphism from $U_1 \times U_2$ to $V_1 \times V_2$.

$\varphi(U_1 \times U_2 \cap N_1 \times N_2) = \varphi_1(U_1 \times N_1) \times \varphi_2(U_2 \times N_2)$.

$= \{x \in V_1 \times V_2 \subset \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \cong \mathbb{R}^{m_1+m_2} \mid x_{n_1+1} = \cdots = x_{m_1} = x_{m_1+n_2+1} = \cdots = x_{m_1+m_2} = 0\}$.

$\Rightarrow N_1 \times N_2$ is a submanifold of $\mathbb{R}^{m_1+m_2}$ of dimension n_1+n_2.

EX8.

(1). Take chart $(\varphi. U_1. V_1)$ near p & $(\psi. U_2. V_2)$ near $F(p)$ s.t. $F(U_1) \subset U_2$.

$d(\psi \circ F \circ \varphi^{-1})(\varphi(p)) = d\psi(F(p)) \circ dF(p) \circ d\varphi^{-1}(\varphi(p)) : T_{\varphi(p)} V_1 \to T_{\psi(F(p))} V_2$

$dF(p)$ is an isomorphism $\Rightarrow d(\psi \circ F \circ \varphi^{-1})(\varphi(p))$ is an isomorphism from $T_{\varphi(p)} V_1 \cong \mathbb{R}^n$ to $T_{\psi(F(p))} V_2 \cong \mathbb{R}^m$.

From Inverse Function thm in the Euclidean space.

$\psi \circ F \circ \varphi^{-1}$ is a diffeomorphism near $\varphi(p)$, that is, $\exists$ neighborhoods of $\varphi(p)$: $X_1 \subset V_1$; of $\psi(F(p))$: $Y_1 \subset V_2$ s.t. $\psi \circ F \circ \varphi^{-1}$ is a diffeomorphism from X_1 to Y_1.

Take $\tilde{U}_1 = \varphi^{-1}(X_1)$. $\tilde{U}_2 = \psi^{-1}(Y_1)$, $F = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi$ is a diffeomorphism from $\tilde{U}_1$ to $\tilde{U}_2$.

(2). If F is an immersion from S^n to $\mathbb{R}^n$, $\forall p \in S^n$.

$dF_p : T_p S^n \to T_p \mathbb{R}^n$ is injective.

$\dim S^n = n = \dim \mathbb{R}^n$. $\Rightarrow dF_p$ is an isomorphism $\forall p \in S^n$.

From (1) F is locally diffeomorphism $\Rightarrow F$ is an open map.

$F(S^n)$ is both closed and open in $\mathbb{R}^n$.

$\Rightarrow F(S^n) = \mathbb{R}^n$. contradiction to compactness of S^n & continuity of F.

EX9.

By definition

$$F^*\alpha(X_1 \dots X_k) := \alpha(F_*(X_1), \dots, F_*(X_k)).$$

$\alpha = uv\,du + 2w\,dv - v\,dw.$ $\quad F^*\alpha \in \Omega^1(\mathbb{R}^2)$

$$F^*\alpha(\tfrac{\partial}{\partial x}) = \alpha(F_*(\tfrac{\partial}{\partial x})) = \alpha(y\tfrac{\partial}{\partial u} + 2x\tfrac{\partial}{\partial v} + 3\tfrac{\partial}{\partial w}).$$

$$= y \cdot x^3y + 2x \cdot 2(3x+y) + 3\cdot(-x^2)$$

$$= x^3y^2 + 9x^2 + 4xy$$

$$F^*\alpha(\tfrac{\partial}{\partial y}) = \alpha(F_*(\tfrac{\partial}{\partial y})) = \alpha(x\tfrac{\partial}{\partial u} + 0\tfrac{\partial}{\partial v} + 1\cdot\tfrac{\partial}{\partial w})$$

$$= x \cdot x^3y + 1\cdot(-x^2)$$

$$= x^4y - x^2.$$

$$\Rightarrow F^*\alpha = (x^3y^2 + 9x^2 + 4xy)\,dx + (x^4y - x^2)\,dy.$$

EX 10.

(1). $F(x,y) = (u,v,w) = (e^y\cos x,\ e^y\sin x,\ e^{-y}) \in \mathbb{R}^3.$

$F_*(\frac{\partial}{\partial x}) = (-\sin x\, e^y,\ \cos x\, e^y,\ 0).$

$F_*(\frac{\partial}{\partial y}) = (\cos x\, e^y,\ \sin x\, e^y,\ -e^{-y})$

If $(u,v,w) \in S_r(0)$ $\quad e^{2y} + e^{-2y} = r^2.$

Note that $e^{2y} + e^{-2y} \geq 2.$ $F^{-1}(S_r(0)) = \phi$ for $r < \sqrt{2}.$

$T_{F(p)}S_r(0)$ is perpendicular to the position vector of $q = F(p).$

that is, $(e^y\cos x,\ e^y\sin x,\ e^{-y}).$

$T_p(\mathbb{R}^2)$ is spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}.$

So to make F transitive to $S_r(0)$ in $\mathbb{R}^3$, only need to require $F_*(\frac{\partial}{\partial x})$ and $F_*(\frac{\partial}{\partial y})$ not to be perpendicular to $(e^y\cos x,\ e^y\sin x,\ e^{-y})$ simultaneously.

$$\langle F_*(\tfrac{\partial}{\partial x}),\ (e^y\cos x,\ e^y\sin x,\ e^{-y})\rangle_{\mathbb{R}^3}$$

$$= -\sin x\cos x\, e^{2y} + \cos x\sin x\, e^{2y} + 0 = 0.$$

$$\Rightarrow \text{We need } 0 \neq \langle F_*(\tfrac{\partial}{\partial y}),\ (e^y\cos x,\ e^y\sin x,\ e^{-y})\rangle_{\mathbb{R}^3}$$

$$= e^{2y} - e^{-2y}$$

$e^{2y} - e^{-2y} \neq 0 \iff y \neq 0 \iff r \neq \sqrt{2}$.

In sum, F is transverse to $S_r(0)$ for all positive $r \neq \sqrt{2}$.

(2). From (1). for $r \neq \sqrt{2}$. $F^{-1}(S_r(0))$ is an embedded submanifold of $\mathbb{R}^2$.

For $r = \sqrt{2}$: $\quad e^{2y} = e^{-2y} \Rightarrow y = 0. \quad x \in \mathbb{R}$.

$F^{-1}(S_{\sqrt{2}}(0)) = \{(x, y) \in \mathbb{R}^2 : y = 0\}$. is exactly an embedded submanifold of $\mathbb{R}^2$.

So $\forall r$ positive, $F^{-1}(S_r(0))$ is an embedded submanifold of $\mathbb{R}^2$.