HOMEWORK FOUR

This homework problem set can be accomplished with the help of references. Every problem (except the Bonus problem) worths 5 point and DO NOT LEAVE ANY PROBLEM BLANK! It is due to FINAL EXAM DAY of this semester.

Exercise 1. Recall that in Hamiltonian Floer homology theory associated to the Hamiltonian system $(M, \omega, H : [0,1] \times M \to \mathbb{R}, J = \{J_t\}_{t \in \mathbb{R}/\mathbb{Z}})$ where (M, ω) is closed symplectic manifold, a Floer cylinder $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ satisfies

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} - \nabla H_t(u) = 0$$

where the gradient ∇ (depending on $t \in \mathbb{R}/\mathbb{Z}$) is taken with respect to the metric $\langle \cdot, \cdot \rangle_t := \omega(\cdot, J_t \cdot)$ (equivalently, $\nabla H_t = J_t X_{H_t}$ where X_{H_t} is the Hamiltonian vector field of H on (M, ω)). Prove that if the energy $E(u) < \infty$, then there exist a sequence of real numbers $\{s_n^+\}_{n \in \mathbb{N}}$ diverging to ∞ and a sequence of real numbers $\{s_n^-\}_{n \in \mathbb{N}}$ diverging to $-\infty$, such that the loops $x_n^\pm := u(s_n^\pm, \cdot)$ converge in C^∞-sense to closed Hamiltonian orbits $x_\pm$ of (M, ω, H, J), respectively. Hint: Use Arzelà-Ascoli Theorem.

Remark 0.1. *Without justifying the C^∞*-convergence *will lose 1 point.*

Exercise 2. Let M be an odd-dimensional manifold equipped with a stable Hamiltonian structure (ω, λ) and J is compatible with (ω, λ). If $u : (\mathbb{R} \times S^1, j) \to (\mathbb{R} \times M, J)$ is a J-holomorphic curve satisfying

$$E_\epsilon(u) < \infty \quad \text{and} \quad \int_{\mathbb{R} \times S^1} u^*((\pi_M)^*\omega) = 0$$

then u is either a constant map or it is biholomorphic to a trivial cylinder over a closed Reeb orbit. Here, recall that $E_\epsilon(\cdot)$ is the “modified energy” defined by

$$E_\epsilon(u) := \sup_{\varphi : \mathbb{R} \to (-\epsilon, \epsilon),\ \varphi' > 0} \int_{\mathbb{R} \times S^1} u^*(\omega + d(\varphi(r)\lambda))$$

and $\pi_M : \mathbb{R} \times M \to \mathbb{R}$ is the projection to M.

Exercise 3. Given a Morse system (M, g, F) where $F : (M, g) \to \mathbb{R}$ is a Morse function, and all the gradient flowlines below are referred to this Morse system. Suppose

$\{u_n\}_{n\in\mathbb{N}}$ is a sequence of a gradient flowlines that converges to a non-constant gradient flowline u in the C^∞_{loc}-sense (i.e, C^∞ over any compact subset of the domain $\mathbb{R}$). Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that the shifted gradient flowlines $\{u_n(\cdot+s_n)\}_{n\in\mathbb{N}}$ also converges to a non-constant gradient flowline $\tilde{u}$ in the C^∞_{loc}-sense. Assume that

$$\text{either } \lim_{s\to-\infty} u(s) = \lim_{s\to-\infty} \tilde{u}(s) \text{ or } \lim_{s\to\infty} u(s) = \lim_{s\to\infty} \tilde{u}(s)$$

i.e., u and $\tilde{u}$ share at least one common asymptotic end. Then prove that the sequence $\{s_n\}_{n\in\mathbb{N}}$ converges to a finite number $s\in\mathbb{R}$ and $\tilde{u}(\cdot) = u(\cdot+s)$.

Bonus (3 points): Derive from Exercise 3 above that if both

$$u^{(1)}\#u^{(2)}\#\cdots\#u^{(m)} \quad\text{and}\quad \tilde{u}^{(1)}\#\tilde{u}^{(2)}\#\cdots\#\tilde{u}^{(m')}$$

serve as a "broken limit" of a sequence of gradient flowlines of the Morse system (M,g,F) (with fixed asymptotic ends $x_\pm\in\mathrm{Crit}(F)$), then $m=m'$ and there exists a sequence of real numbers $\{s^{(i)}\}_{i\in\{1,\ldots,m\}}$ such that $\tilde{u}^{(i)}(\cdot) = u^{(i)}(\cdot+s^{(i)})$.