2024秋微分流形第三次作业解答 2024.10.

1. $\Lambda^k V^*$的基.

自然嵌入: $\Lambda^k V^* \hookrightarrow V^{*,\otimes k}$.

<u>Recall</u> 课上定义的反对称化算子 Alt

$$Alt(T) = \frac{1}{k!}\sum_{\sigma\in S_k} sgn(\sigma)(\sigma T),$$

其中 $T\in V^{*,\otimes k}$.

<u>Fact</u> $\Lambda^k V^* := \{T\in V^{*,\otimes k} : \sigma T = sgn(\sigma)T\} = Im(Alt)$.

因此: 由 $V^{*,\otimes k} = span\{e^{i_1}\otimes\cdots\otimes e^{i_k},\ 1\le i_1,\cdots,i_k\le n\}$ 得到:

$$\Lambda^k V^* = span\{\underbrace{Alt(e^{i_1}\otimes\cdots\otimes e^{i_k})},\ 1\le i_1,\cdots,i_k\le n\}$$

外代数的定义. $= k!\, e^{i_1}\wedge\cdots\wedge e^{i_k}$

$$= span\{e^{i_1}\wedge\cdots\wedge e^{i_k},\ 1\le i_1,\cdots,i_k\le n\}.$$

故须考虑 $span\{e^{i_1}\wedge\cdots\wedge e^{i_k},\ 1\le i_1,\cdots,i_k\le n\}$ 的一组基是

$$\{e^{i_1}\wedge\cdots\wedge e^{i_k},\ 1\le i_1<\cdots<i_k\le n\}$$

即可. 这是易于验证的. #.

2. 证明: Hodge $*$算子满足 $*^2=\pm\mathrm{Id}$.

利用Ex1的结论，只需在 $\Lambda^k V$ 上的一组基验证即可.

对 $e_{i_1}\wedge\cdots\wedge e_{i_k}$, $1\le i_1<\cdots<i_k\le n$, 取 $\{\tilde{e}_i\}_{i=1}^n=\pm\{e_i\}_{i=1}^n$ 以使:

$$\tilde{e}_1=e_{i_1},\ \cdots,\ \tilde{e}_k=e_{i_k},\quad \underline{\tilde{e}_1\wedge\cdots\wedge\tilde{e}_n=e_1\wedge\cdots\wedge e_n}.$$

则$*$算子的定义: 这时内积有符号差，因此对偶作用时也有符号之差: $\forall v_{k+1},\cdots,v_n\in V$

$$\begin{aligned}
&(\iota_{e_1\wedge\cdots\wedge e_k}e^1\wedge\cdots\wedge e^n)(v_{k+1},\cdots,v_n)\\
&=e^1\wedge\cdots\wedge e^n(e_1,\cdots,e_k,v_{k+1},\cdots,v_n)\\
&=\underline{(-1)^{\min\{k,p\}}}\,e^{k+1}\wedge\cdots\wedge e^n(v_{k+1},\cdots,v_n)
\end{aligned}$$

注意: $\underline{e^i(e_j)=\begin{cases}\delta_{ij}, & p+1\le i\le n\\ -\delta_{ij}, & 1\le i\le p\end{cases}}$ (内积的负惯性指数为p)

故 $\iota_{e_1\wedge\cdots\wedge e_k}e^1\wedge\cdots\wedge e^n=(-1)^{\min\{k,p\}}e^{k+1}\wedge\cdots\wedge e^n$

再由$*$算子的定义:

$$\begin{tikzcd}
\Lambda^k V \arrow[rr, "*"] \arrow[dr, "\iota"'] & & \Lambda^{n-k}V \\
& \Lambda^{n-k}V^* \arrow[ur, "\text{duality}"'] &
\end{tikzcd}$$

$e_1\wedge\cdots\wedge e_k$ $\qquad (-1)^{\min\{k,p\}}e_{k+1}\wedge\cdots\wedge e_n$

$(-1)^{\min\{k,p\}}e^{k+1}\wedge\cdots\wedge e^n$

这个映射依赖于 $\Lambda^n V$ 元素的选取!

从而：$*(e_1\wedge\cdots\wedge e_k)=(-1)^{\min\{k,p\}}e_{k+1}\wedge\cdots\wedge e_n$

类似地，可以计算：

$$(\iota_{e_{k+1}\wedge\cdots\wedge e_n}e^1\wedge\cdots\wedge e^n)(v_1,\cdots,v_k)$$

$$=e^1\wedge\cdots\wedge e^n(e_{k+1},\cdots,e_n,v_1,\cdots,v_k)$$

$$=(-1)^{k(n-k)}e^{k+1}\wedge\cdots\wedge e^n\wedge e^1\wedge\cdots\wedge e^k(e_{k+1},\cdots,e_n,v_1,\cdots,v_k)$$

$$=(-1)^{k(n-k)}(-1)^{\max\{0,p-k\}}e^1\wedge\cdots\wedge e^k(v_1,\cdots,v_k)$$

故 $*(e_{k+1}\wedge\cdots\wedge e_n)=(-1)^{k(n-k)}(-1)^{\max\{0,p-k\}}e_1\wedge\cdots\wedge e_k$

从而：$*^2(e_1\wedge\cdots\wedge e_k)=(-1)^{k(n-k)}(-1)^{\min\{p,k\}+\max\{0,p-k\}}e_1\wedge\cdots\wedge e_k$

$$=(-1)^{k(n-k)+p}e_1\wedge\cdots\wedge e_k.$$

因 $\tilde{e}_1\wedge\cdots\wedge\tilde{e}_n=e_1\wedge\cdots\wedge e_n$，故：

$$*^2(\tilde{e}_1\wedge\cdots\wedge\tilde{e}_k)=(-1)^{k(n-k)+p}\tilde{e}_1\wedge\cdots\wedge\tilde{e}_k$$

即 $*^2(e_{i_1}\wedge\cdots\wedge e_{i_k})=(-1)^{k(n-k)+p}e_{i_1}\wedge\cdots\wedge e_{i_k}$.

这就验证了 $*^2=(-1)^{k(n-k)+p}\mathrm{Id}$ #.

<u>Rmk</u>. Hodge $*$ 算子的另一等价形式可以表示为：指定 $\omega\in\wedge^n V^*$，

$\alpha\in\wedge^k V$，$*\alpha\in\wedge^{n-k}V$ 满足：$\omega(\alpha\wedge*\alpha)=1$.

3. 直接计算即可.

Rmk 这里有一个小问题：对向量场的导数是什么？$\frac{\partial}{\partial s}X_{s,t}$，$\frac{\partial}{\partial t}Y_{s,t}$ 是什么？为避免问题，下面均认为这些都在局部坐标下存在.

$\forall p\in M$，取 p 附近的坐标卡 (f,U) 和 $\varphi_{s,t}(p)$ 附近的坐标卡 (g,V) 使得 $\varphi_{s,t}(U)\subset V$. 记 $x_0=f(p)$，x,y 分别为 (f,U)，(g,V) 的坐标.

$$\frac{\partial}{\partial t}\varphi_{s,t}(p)=\frac{\partial}{\partial t}(g\circ\varphi_{s,t}\circ f^{-1})(x_0)$$
$$=X_{s,t}\circ\varphi_{s,t}(p)$$
$$=(X^1_{s,t},\cdots,X^n_{s,t})\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)$$

两边同时对 s 求导数，则：

类似地，$\frac{\partial}{\partial s}(g\circ\varphi_{s,t}\circ f^{-1})(x_0)$
$=Y_{s,t}\circ\varphi_{s,t}(p)$

$$\frac{\partial}{\partial s}\left[(X^1_{s,t},\cdots,X^n_{s,t})\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)\right]$$
$$=\left(\frac{\partial}{\partial s}X^1_{s,t},\cdots,\frac{\partial}{\partial s}X^n_{s,t}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)$$
$$+\sum_{i=1}^n\Big(\left(\frac{\partial}{\partial y^i}X^1_{s,t}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x)\frac{\partial}{\partial s}(g\circ\varphi_{s,t}\circ f^{-1})^i(x_0),\cdots,$$
$$\left(\frac{\partial}{\partial y^i}X^n_{s,t}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x)\frac{\partial}{\partial s}(g\circ\varphi_{s,t}\circ f^{-1})^i(x_0)\Big)$$
$$=\left(\frac{\partial}{\partial s}X_{s,t}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)+\sum_{i=1}^n\left(Y^i_{s,t}\frac{\partial X^1_{s,t}}{\partial y^i},\cdots,Y^i_{s,t}\frac{\partial X^n_{s,t}}{\partial y^i}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)$$
$$=\frac{\partial^2}{\partial s\partial t}(g\circ\varphi_{s,t}\circ f^{-1})(x_0)$$

类似地，有：

$$
\begin{aligned}
&\frac{\partial^2}{\partial t\partial s}(g\circ\varphi_{s,t}\circ f^{-1})(x_0)\\
=&\left(\frac{\partial}{\partial t}Y_{s,t}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)\\
&+\sum_{i=1}^{n}\left(X^i_{s,t}\frac{\partial Y^1_{s,t}}{\partial y^i},\cdots,X^i_{s,t}\frac{\partial Y^n_{s,t}}{\partial y^i}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0).
\end{aligned}
$$

相减得到：

$$
\begin{aligned}
&\left(\frac{\partial}{\partial s}X_{s,t}-\frac{\partial}{\partial t}Y_{s,t}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)\\
=&\underbrace{\left(\frac{\partial^2}{\partial s\partial t}-\frac{\partial^2}{\partial t\partial s}\right)}_{\text{光滑，故为0}}(g\circ\varphi_{s,t}\circ f^{-1})(x_0)\\
&+\sum_{i=1}^{n}\left(X^i_{s,t}\frac{\partial Y^1_{s,t}}{\partial y^i}-Y^i_{s,t}\frac{\partial X^1_{s,t}}{\partial y^i},\cdots,X^i_{s,t}\frac{\partial Y^n_{s,t}}{\partial y^i}-Y^i_{s,t}\frac{\partial X^n_{s,t}}{\partial y^i}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)\\
=&\sum_{i=1}^{n}\left(X^i_{s,t}\frac{\partial Y^1_{s,t}}{\partial y^i}-Y^i_{s,t}\frac{\partial X^1_{s,t}}{\partial y^i},\cdots,X^i_{s,t}\frac{\partial Y^n_{s,t}}{\partial y^i}-Y^i_{s,t}\frac{\partial X^n_{s,t}}{\partial y^i}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)\\
=&\underbrace{\left(D_{X_{s,t}}Y^1_{s,t}-D_{Y_{s,t}}X^1_{s,t},\cdots,D_{X_{s,t}}Y^n_{s,t}-D_{Y_{s,t}}X^n_{s,t}\right)}_{\text{课上给出的Lie括号定义}}\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)\\
=&[X_{s,t},Y_{s,t}]\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0).
\end{aligned}
$$

从而：$\frac{\partial}{\partial s}X_{s,t}-\frac{\partial}{\partial t}Y_{s,t}=[X_{s,t},Y_{s,t}]$. #

Rmk. 课上给出了Lie括号的定义是在局部坐标下的，

但实际上Lie括号的定义不依赖于坐标卡.

4. 证明 $L_X Y=[X,Y]$, $\forall X,Y\in\Gamma(TM)$.

下面利用 Cartan's Magic Formula 给出证明.

$\forall \omega\in\Omega^1(M)$, 先导出一个公式:

$$\omega(L_XY)=X(\omega(Y))-(L_X\omega)(Y)$$
$$=X(\omega(Y))-(\iota_Xd\omega)(Y)-(d\iota_X\omega)(Y)$$
$$=X(\omega(Y))-d\omega(X,Y)-d(\omega(X))(Y)$$
$$=X(\omega(Y))-Y(\omega(X))-d\omega(X,Y)=\omega([X,Y])$$

从而 $L_XY=[X,Y]$. (因 $\Omega^1(M)=(TM)^*$) #

Rmk1 最直接的办法是在局部坐标下计算. 见 GTM218, P229.

Rmk2 注意 Cartan's Magic Formula 的证明是不需要 $L_XY=[X,Y]$ 的

因此不会循环论证.

Rmk3 用到了 Leibniz 公式: 设 $\omega\in\Omega^1(M)$, $X,Y\in\Gamma(TM)$,

$$(L_X\omega)(Y)=X(\omega(Y))-\omega(L_XY).$$

证明类似. (利用定义即可). 证明在下一页.

Rmk4 有同学问到了 $[X,Y]$ 的定义.

Recall 课上证明了 $(L_X df)(Y) = Y(Xf)$

由此，将有：$\forall f \in C^\infty(M)$,

$$0 = d^2 f(Y,X) = Y(df(X)) - X(df(Y)) - df([X,Y]),$$

故 $[X,Y](f) = X(Yf) - Y(Xf)$, $\forall f \in C^\infty(M)$.

Pf of Rmk3:
$$(L_X\omega)(Y)_p = \lim_{t\to 0}\frac{(\varphi_t^X)_p^*\omega_{\varphi_t^X(p)}(Y_p) - \omega_p(Y_p)}{t}$$

$$= \lim_{t\to 0}\frac{\omega_{\varphi_t^X(p)}((d\varphi_t^X)_p(Y_p)) - \omega_p(Y_p)}{t}$$

$$= \lim_{t\to 0}\frac{\omega_{\varphi_t^X(p)}(Y_{\varphi_t^X(p)}) - \omega_p(Y_p)}{t} + \lim_{t\to 0}\frac{\omega_{\varphi_t^X(p)}((d\varphi_t^X)_p(Y_p) - Y_{\varphi_t^X(p)})}{t}$$

$$= \frac{(\omega(Y))_{\varphi_t^X(p)} - (\omega(Y))_p}{t} \to X(\omega(Y))_p,\ t\to 0$$

$$= X(\omega(Y))_p + \lim_{t\to 0}\frac{(\varphi_t^X)_p^*\omega_{\varphi_t^X(p)}(Y_p - (d\varphi_{-t}^X)^*_{\varphi_t^X(p)}(Y_{\varphi_t^X(p)}))}{t}$$

$$Y_p - (d\varphi_{-t}^X)^*_{\varphi_t^X(p)}(Y_{\varphi_t^X(p)}) = -t(L_XY)_p + o(t)$$

Lie导数定义.

$$= X(\omega(Y))_p - \lim_{t\to 0}(\varphi_t^X)_p^*\omega_{\varphi_t^X(p)}((L_XY))_p - \lim_{t\to 0}\frac{(\varphi_t^X)_p^*\omega_{\varphi_t^X(p)}(o(t))}{t}$$

$$\lim_{t\to 0}\frac{(\varphi_t^X)_p^*\omega_{\varphi_t^X(p)}(o(t))}{t} = 0$$

$$(\varphi_t^X)_p^*\omega_{\varphi_t^X(p)} \xrightarrow{t\to 0} \omega$$

$$= X(\omega(Y))_p - \omega_p((L_XY)_p),\ \forall p \in M$$

故 $(L_X\omega)(Y) = X(\omega(Y)) - \omega(L_XY)$. #

5. Poisson括号满足Jacobi恒等式.

(有时候把$[X,Y]$叫作Lie括号, 本题定义的$\{\cdot,\cdot\}$叫Poisson括号)

<u>Recall</u> 课上证明了: $L_{X_H}\omega=0$ & $X_H(H)=0$

$$\{\{H,G\},F\}=\omega(X_{\{H,G\}},X_F)=-d\{H,G\}(X_F)$$

$$=-X_F(\{H,G\})$$

$$=-X_F(\omega(X_H,X_G))$$

(Leibniz) $$=-(L_{X_F}\omega)(X_H,X_G)+\omega(L_{X_F}X_H,X_G)+\omega(X_H,L_{X_F}X_G)$$

$$=\omega([X_F,X_H],X_G)+\omega(X_H,[X_F,X_G]).$$

类似地, $$\{\{G,F\},H\}=\omega([X_H,X_G],X_F)+\omega(X_G,[X_H,X_F])$$

$$=\omega([X_H,X_G],X_F)+\omega([X_F,X_H],X_G),$$

$$\{\{F,H\},G\}=\omega([X_G,X_F],X_H)+\omega(X_F,[X_G,X_H])$$

$$=\omega(X_H,[X_F,X_G])+\omega([X_H,X_G],X_F),$$

故 $\{\{H,G\},F\}+\{\{G,F\},H\}+\{\{F,H\},G\}$

$$=2\omega([X_F,X_H],X_G)+2\omega(X_H,[X_F,X_G])+2\omega([X_H,X_G],X_F)$$

$$=2[X_F,X_H](G)-2[X_F,X_G](H)+2[X_H,X_G](F)$$

$$=2X_F(X_H(G))-2X_H(X_F(G))-2X_F(X_G(H))$$

$$+2X_G(X_F(H))+2X_H(X_G(F))-2X_G(X_H(F))$$

$d\omega=0$, 故:

$$0=d\omega(X_H,X_G,X_F)$$
$$=X_H(\omega(X_G,X_F))-X_G(\omega(X_H,X_F))+X_F(\omega(X_H,X_G))$$
$$-\omega([X_H,X_G],X_F)+\omega([X_H,X_F],X_G)-\omega([X_G,X_F],X_H)$$
$$=X_H(-X_F(G))-X_G(-X_F(H))+X_F(-X_G(H))$$
$$-(-[X_H,X_G](F))+(-[X_H,X_F](G))-(-[X_G,X_F](H))$$
$$=X_G(X_F(H))-X_F(X_G(H))-X_H(X_F(G))$$
$$+[X_H,X_G](F)-[X_H,X_F](G)+[X_G,X_F](H)$$
$$=2X_G(X_F(H))-2X_F(X_G(H))-\cancel{X_H}(X_F(G))+X_F(X_H(G))$$
$$+X_H(X_G(F))-X_G(X_H(F))-X_H(X_F(G))$$

且 $X_H(G)=dG(X_H)=-\omega(X_G,X_H)=\omega(X_H,X_G)=-dH(X_G)=-X_G(H)$

故上式 $\Leftrightarrow 0=3X_G(X_F(H))-3X_F(X_G(H))-3X_H(X_F(G))$

$$\Leftrightarrow X_G(X_F(H))=X_F(X_G(H))+X_H(X_F(G)) \quad (☆)$$

这样得到一组恒等式. 下面代入计算.

$$\{\{H,G\},F\}+\{\{G,F\},H\}+\{\{F,H\},G\}$$

$$=2X_F(X_H(G))-2X_H(X_F(G))-2X_F(X_G(H))$$
$$+2X_G(X_F(H))+2X_H(X_G(F))-2X_G(X_H(F))$$

$$=4X_F(X_H(G))-4X_H(X_F(G))-4X_G(X_H(F))$$

$$=4X_G(X_H(F))-4X_H(X_F(G))-4X_F(X_G(H))$$

$=0$，因(☆)成立. #

为什么？

Rmk. (M^{2n},ω) 称为辛流形 (Symplectic Manifolds)，$\dim M$ 为偶数.

Darboux定理：在 M 的任一点处，$\exists$ 坐标卡 $((x,y),U)$ 使得 $\mathbb{R}^n\times\mathbb{R}^n\simeq\mathbb{R}^{2n}$.

$$\omega=\sum_{i=1}^{n}dx^i\wedge dy^i$$

可以算出 X_H：在上述的坐标卡下，有

$$X_H=\sum_{i=1}^{n}\left(-\frac{\partial H}{\partial y^i}\frac{\partial}{\partial x^i}+\frac{\partial H}{\partial x^i}\frac{\partial}{\partial y^i}\right),$$

这样，本题定义的 $\{\cdot,\cdot\}$ 可以表示为：

$$\{H,G\}=\omega(X_H,X_G)=X_H(G)$$
$$=\sum_{i=1}^{n}\left(\frac{\partial H}{\partial x^i}\frac{\partial G}{\partial y^i}-\frac{\partial H}{\partial y^i}\frac{\partial G}{\partial x^i}\right).$$

6、GTM218, Ex 8.10.

$\varphi(x,y)=(xy,\dfrac{y}{x})$，不妨记坐标卡为 $\varphi:(x,y)\mapsto(u,v)=(\varphi^1,\varphi^2)$

$$J\varphi=\begin{pmatrix}\dfrac{\partial\varphi^1}{\partial x} & \dfrac{\partial\varphi^1}{\partial y}\\ \dfrac{\partial\varphi^2}{\partial x} & \dfrac{\partial\varphi^2}{\partial y}\end{pmatrix}=\begin{pmatrix} y & x\\ -\dfrac{y}{x^2} & \dfrac{1}{x}\end{pmatrix},\quad (x,y)\in\mathbb{R}_+^2$$

进而 $\varphi_*X=d\varphi(X)=x\left(\dfrac{\partial\varphi^1}{\partial x}\dfrac{\partial}{\partial u}+\dfrac{\partial\varphi^2}{\partial x}\dfrac{\partial}{\partial v}\right)+y\left(\dfrac{\partial\varphi^1}{\partial y}\dfrac{\partial}{\partial u}+\dfrac{\partial\varphi^2}{\partial y}\dfrac{\partial}{\partial v}\right)$

$$=xy\frac{\partial}{\partial u}-\frac{y}{x}\frac{\partial}{\partial v}+xy\frac{\partial}{\partial u}+\frac{y}{x}\frac{\partial}{\partial v}$$

$$=2xy\frac{\partial}{\partial u}=2u\frac{\partial}{\partial u},$$

$$\varphi_*Y=d\varphi(Y)=y\left(\frac{\partial\varphi^1}{\partial x}\frac{\partial}{\partial u}+\frac{\partial\varphi^2}{\partial x}\frac{\partial}{\partial v}\right)$$

$$=y^2\frac{\partial}{\partial u}-\frac{y^2}{x^2}\frac{\partial}{\partial v}$$

$$=uv\frac{\partial}{\partial u}-v^2\frac{\partial}{\partial v}.$$

#

7、GTM218, Ex 14.7(a)

$\varphi:(x,y)\mapsto(u,v)=(\varphi^1,\varphi^2)=(xy,e^{-y})$

在 (u,v) 坐标下，$\alpha=u\,dv$，于是

$$\varphi^*\alpha=u\,\varphi^*dv=xy\cdot(-e^{-y}dy)=-xye^{-y}dy.$$

知：$\varphi^*(d\alpha)=\varphi^*(du\wedge dv)=\varphi^*(du)\wedge\varphi^*(dv)=-ye^{-y}dx\wedge dy$,

$d(\varphi^*\alpha)=d(-xye^{-y}dy)=-ye^{-y}dx\wedge dy=\varphi^*(d\alpha)$.

#

8. GTM 218, Ex 9.17.

(1) 这结论需要用到ODE. 因 $X(p)\neq 0$, 取 p 处的一个坐标卡 $(\tilde{\varphi}, U)$.

不妨设 $\tilde{\varphi}(p)=0$, $X(p)=\frac{\partial}{\partial x^1}\Big|_0$. 适当缩小 U, 使得:

$$X|_U := \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i}, \quad f^1|_U \neq 0.$$

给定初值 $(u^2, \cdots, u^n)$, 解ODE问题:

$$\begin{cases} \dfrac{d}{dt} y^i(t; u^2, \cdots, u^n) = \dfrac{f^i(t, y^2, \cdots, y^n)}{f^1(t, y^2, \cdots, y^n)}, & i\geq 2 \\ y^i(0; u^2, \cdots, u^n) = u^i. \end{cases}$$

解的存在性, 对初值光滑依赖性由ODE理论得到, 定义 φ:

$$\begin{cases} x^1 = u^1, \\ x^i = f^i(u^1; u^2, \cdots, u^n), & i \geq 2, \end{cases}$$

则 $J\varphi|_0 = Id$, 故 φ 为局部的微分同胚. 缩小 U 使得 $\varphi: V \to U$ 为微分同胚. 于是:

$$\begin{aligned} (f^1\circ\varphi)\varphi_*\left(\frac{\partial}{\partial u^1}\right) &= (f^1\circ\varphi)\sum_{i=1}^{n} \frac{\partial x^i}{\partial u^1}\left(\frac{\partial}{\partial x^i}\circ\varphi\right) \\ &= (f^1\circ\varphi)\left(\frac{\partial}{\partial x^1}\circ\varphi\right) + (f^1\circ\varphi)\sum_{i=2}^{n}\frac{f^i\circ\varphi}{f^1\circ\varphi}\left(\frac{\partial}{\partial x^i}\circ\varphi\right) \\ &= X\circ\varphi, \end{aligned}$$

这距离想要的结果只差一步.

再取微分同胚ψ:

$$\begin{cases} v^1 = \int_0^{u^1} \dfrac{dt}{f'\circ\varphi(t,u^2,\cdots,u^n)} \\ v^i = u^i, \quad i \geq 2 \end{cases}$$

可以取$\psi: V \to W$为微分同胚. 于是

$$\psi_*^{-1}\left(\frac{\partial}{\partial v^1}\right) = \sum_{i=1}^{n} \frac{\partial u^i}{\partial v^1}\left(\frac{\partial}{\partial u^i}\circ\psi^{-1}\right)$$

$$= \frac{\partial u^1}{\partial v^1}\left(\frac{\partial}{\partial u^1}\circ\psi^{-1}\right) = (f'\circ\varphi\circ\psi^{-1})\left(\frac{\partial}{\partial u^1}\circ\psi^{-1}\right)$$

$$= \varphi_*^{-1}(X)\circ\varphi\circ\psi^{-1},$$

从而在这个坐标卡下, $X = \frac{\partial}{\partial v^1}$.

(2) Check $[X_1, X_2] = -X_3$, $[X_1, X_3] = X_2$, $[X_2, X_3] = -X_1$.

注意到: $zV_1 + xV_2 = \cancel{xz\frac{\partial}{\partial y}} - yz\frac{\partial}{\partial x} + xy\frac{\partial}{\partial z} - \cancel{xz\frac{\partial}{\partial y}}$

$$= y\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right) = -yV_3$$

这表明$\{X_1, X_2, X_3\}$是线性相关的(在$(1,0,0)$附近)

因此不存在坐标卡使得$X_i = \partial/\partial x^i$.

9. Killing 向量场.

首先导出 X 满足的条件. 记 $\mathbb{R}^3$ 上的欧氏联络为 D, 则:

$$X(g(Y,Z))=g(D_XY,Z)+g(Y,D_XZ)$$

对 $\forall X,Y,Z\in\Gamma(TM)$. 于是:

$$\begin{aligned}(L_Xg)(Y,Z)=0&=X(g(Y,Z))-g(L_XY,Z)-g(Y,L_XZ)\\&=g(D_XY-L_XY,Z)+g(Y,D_XZ-L_XZ).\end{aligned}$$

且 $D_XY-D_YX=[X,Y]$, $L_XY=[X,Y]$, 故.

$$g(D_YX,Z)+g(Y,D_ZX)=0$$

这就是 Killing 向量场的另一等价定义.

在 $\mathbb{R}^3$ 的坐标下, 记 $X=X^i\partial_i$, 则:

$$\begin{aligned}&g(D_{\partial_i}X,\partial_j)+g(\partial_i,D_{\partial_j}X)=0\\&=g(\partial_iX^k\partial_k,\partial_j)+g(\partial_i,\partial_jX^k\partial_k)\\&=\partial_iX^kg_{jk}+\partial_jX^kg_{ik}=0\end{aligned}$$

记 $G=(g_{ij})_{n\times n}$, 则 $G=\mathrm{diag}(1,1,-1)$; 记 $A=(\partial_iX^j)_{n\times n}$,

则 $AG+GA^T=0=(AG)+(AG)^T$

展开写为：$\begin{pmatrix} \partial_1 X^1 & \partial_1 X^2 & -\partial_1 X^3 \\ \partial_2 X^1 & \partial_2 X^2 & -\partial_2 X^3 \\ \partial_3 X^1 & \partial_3 X^2 & -\partial_3 X^3 \end{pmatrix}$ 为反对称的.

$$\left.\begin{aligned} &\text{故 } \partial_1 X^1 = \partial_2 X^2 = \partial_3 X^3 = 0, \\ &\partial_1 X^2 + \partial_2 X^1 = 0 \quad ① \\ &\partial_1 X^3 = \partial_3 X^1 \quad ② \\ &\partial_2 X^3 = \partial_3 X^2 \quad ③ \end{aligned}\right\}$$

设 $X^1 = f(y,z)$

$X^2 = h(x,z)$

$X^3 = k(x,y)$

求解方程组.

$$① \Longleftrightarrow \frac{\partial h}{\partial x} + \frac{\partial f}{\partial y} = 0 \Rightarrow \frac{\partial^2 h}{\partial x^2} = 0,\ \frac{\partial^2 f}{\partial y^2} = 0$$

$$② \Longleftrightarrow \frac{\partial k}{\partial x} = \frac{\partial f}{\partial z} \Rightarrow \frac{\partial^2 k}{\partial x^2} = 0,\ \frac{\partial^2 f}{\partial z^2} = 0$$

$$③ \Longleftrightarrow \frac{\partial h}{\partial z} = \frac{\partial k}{\partial y} \Rightarrow \frac{\partial^2 h}{\partial z^2} = 0,\ \frac{\partial^2 k}{\partial y^2} = 0$$

$$\left.\begin{aligned} &\text{以及：} \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 k}{\partial x \partial y} = \frac{\partial^2 h}{\partial x \partial z} \quad (② + ③) \\ &\frac{\partial^2 f}{\partial y \partial z} = -\frac{\partial^2 h}{\partial x \partial z} \quad (①) \end{aligned}\right\} \text{3项均为0.}$$

从而 f, h, k 均为线性函数，

$$\begin{cases} f = ay + bz + d_1, \\ h = -ax + cz + d_2, \\ k = bx + cy + d_3, \end{cases} \qquad a, b, c, d_1, d_2, d_3 \in \mathbb{R}$$

(1) 所有的 Killing 向量场均具有形式：

$$X = a(y,-x,0)+b(z,0,x)+c(0,z,y)+(d_1,d_2,d_3),$$

其中 $a,b,c,d_1,d_2,d_3 \in \mathbb{R}$；

(2). 直接计算，分别记 $X_1=(y,-x,0)$，$X_2=(z,0,x)$，

$X_3=(0,z,y)$，则：

$$[X_1,X_2]=y\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial x}+x\frac{\partial}{\partial z}\right)-x\frac{\partial}{\partial y}\left(z\frac{\partial}{\partial x}+x\frac{\partial}{\partial z}\right)$$

$$-z\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)-x\frac{\partial}{\partial z}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)$$

$$=y\frac{\partial}{\partial z}+z\frac{\partial}{\partial y}=X_3,$$

$$[X_1,X_3]=y\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}\right)-x\frac{\partial}{\partial y}\left(z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}\right)$$

$$-z\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)-y\frac{\partial}{\partial z}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)$$

$$=-x\frac{\partial}{\partial z}-z\frac{\partial}{\partial x}=-X_2,$$

$$[X_2,X_3]=z\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}\right)+x\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial y}+y\frac{\partial}{\partial y}\right)$$

$$-z\frac{\partial}{\partial y}\left(z\frac{\partial}{\partial x}+x\frac{\partial}{\partial z}\right)-y\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial x}+x\frac{\partial}{\partial z}\right)$$

$$=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}=-X_1,$$

Lie 括号运算在 $\mathrm{span}\{X_1,X_2,X_3\}$ 下是封闭的. #

因此：若 X,Y 为 Killing 场，则 $[X,Y]$ 也是 Killing 场

10. 切触形式 (Contact Form)

(1) 取 M 的一个坐标卡图册 $\{(\varphi_\beta, U_\beta)\}$, $\varphi_\beta: U_\beta \xrightarrow{\text{diff}} \mathbb{R}^n$.

给 TM 赋予一个 Riemann 度量 g, 定义 $A: \Gamma(TM) \to \Gamma(TM)$,

$$g(AX, Y) := d\alpha(X, Y),\ \forall X, Y \in \Gamma(TM),$$

实际上 $A \in \Gamma(T^{(1,1)}M)$. 且: $g(AX, Y) = -g(X, AY)$

实矩阵的相似理论 $\Rightarrow$ $A|_{U_\beta}$ 有唯一的一组 0 特征子空间.

$\alpha \wedge d\alpha$ 处处不消失 $\Rightarrow$ $d\alpha$ 处处不为 0

故 A 处处不恒为 0

取 $A|_{U_\beta}$ 的一个 0 特征方向 $R_\beta \in \Gamma(TU_\beta)$ 满足 $\alpha|_{U_\beta}(R_\beta) = 1$

Fact α 在 A 的 0 特征子空间上不为 0.

若不然, 取 A 的一个 0 特征方向 $R \neq 0$, $\alpha(R) = 0$, 则:

$$\iota_R(\alpha \wedge d\alpha) = (\iota_R \alpha) \wedge d\alpha + \alpha \wedge (\iota_R d\alpha)$$

$$= \alpha(R) d\alpha + \alpha \wedge (\iota_R d\alpha) = 0$$

这与 $\alpha \wedge d\alpha$ 处处不消失矛盾!

定义 $R_\alpha \in \Gamma(TM)$: $R_\alpha|_{U_\beta} = R_\beta$

Check 定义合理.

若 $U_\beta \cap U_\gamma \neq \emptyset$，在 $(\varphi_\beta, U_\beta \cap U_\gamma)$ 上，设 $R_\gamma = \lambda_\gamma^\beta R_\beta$

则：$\alpha|_{U_\beta}(R_\gamma) = 1 = \alpha|_{U_\beta}(\lambda_\gamma^\beta R_\beta) = \lambda_\gamma^\beta$

故 $R_\gamma = R_\beta$. 从而定义合理.

由此定义的 R_α 就满足 $\iota_{R_\alpha} d\alpha = 0$ 且 $\alpha(R_\alpha) = 1$.

(2) $\iota_{R_\alpha} d\alpha = 0$, $\iota_{R_\alpha}\alpha = \alpha(R_\alpha) = 1$. 故：

$$L_{R_\alpha}\alpha = d(\iota_{R_\alpha}\alpha) + \iota_{R_\alpha} d\alpha = 0.$$

(3) $\mathbb{R}^3$ 上的一个切触形式 $\alpha = dx + y\,dz$

Check：$\alpha \wedge d\alpha = dx \wedge dy \wedge dz$ 处处不消失

$(d\alpha = dy \wedge dz)$

对应的 $R_\alpha = \frac{\partial}{\partial x}$. $\alpha(R_\alpha) = dx(R_\alpha) + y\,dz(R_\alpha) = 1$,

$\iota_{R_\alpha} d\alpha = \iota_{\frac{\partial}{\partial x}} dy \wedge dz = 0$.

Rmk 实际上 M 是可定向的，因 $\alpha \wedge d\alpha$ 处处不消失.

见 GTM218：$\exists \omega \in \Omega^n(M)$, $n = \dim(M)$, ω 处处不消失

$\Longleftrightarrow$ M 可定向