2024秋微分流形第三次作业解答 2024.10.

1. $\Lambda^k V^*$的基.

自然嵌入：$\Lambda^k V^* \hookrightarrow V^{*,\otimes k}$.

Recall 课上定义的反对称化算子Alt

$$\mathrm{Alt}(T) = \frac{1}{k!}\sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)(\sigma T),$$

其中 $T \in V^{*,\otimes k}$.

Fact $\Lambda^k V^* := \{T\in V^{*,\otimes k} : \sigma T = \mathrm{sgn}(\sigma) T\} = \mathrm{Im}(\mathrm{Alt})$.

因此：由 $V^{*,\otimes k} = \mathrm{span}\{e^{i_1}\otimes\cdots\otimes e^{i_k}, 1\le i_1,\cdots,i_k\le n\}$ 得到：

$$\Lambda^k V^* = \mathrm{span}\{\underbrace{\mathrm{Alt}(e^{i_1}\otimes\cdots\otimes e^{i_k})}_{\text{外代数的定义}\ = k!\, e^{i_1}\wedge\cdots\wedge e^{i_k}}, 1\le i_1,\cdots,i_k\le n\}$$

$$= \mathrm{span}\{e^{i_1}\wedge\cdots\wedge e^{i_k}, 1\le i_1,\cdots,i_k\le n\}.$$

故须考虑 $\mathrm{span}\{e^{i_1}\wedge\cdots\wedge e^{i_k}, 1\le i_1,\cdots,i_k\le n\}$ 的一组基.

$$\{e^{i_1}\wedge\cdots\wedge e^{i_k}, 1\le i_1<\cdots<i_k\le n\}$$

即可. 这是易于验证的. #.

2. 证明: Hodge $*$ 算子满足 $*^2=\pm \mathrm{Id}$.

利用Ex1的结论, 只需在 $\Lambda^k V$ 上的一组基验证即可.

对 $e_{i_1}\wedge\cdots\wedge e_{i_k}$, $1\leq i_1<\cdots<i_k\leq n$, 取 $\{\tilde{e}_i\}_{i=1}^n=\{\pm e_i\}_{i=1}^n$ 以及:

$$\tilde{e}_1=e_{i_1},\cdots,\tilde{e}_k=e_{i_k}.\ \tilde{e}_1\wedge\cdots\wedge\tilde{e}_n=e_1\wedge\cdots\wedge e_n.$$

则 $*$ 算子的定义: 这时内积有符号差, 因此对偶作用时也有符号之差: $\forall v_{k+1},\cdots,v_n\in V$

$$(\iota_{e_1\wedge\cdots\wedge e_k}e^1\wedge\cdots\wedge e^n)(v_{k+1},\cdots,v_n)$$

$$=e^1\wedge\cdots\wedge e^n(e_1,\cdots,e_k,v_{k+1},\cdots,v_n)$$

$$=(-1)^{\min\{k,p\}}e^{k+1}\wedge\cdots\wedge e^n(v_{k+1},\cdots,v_n)$$

注意: $e^i(e_j)=\begin{cases}\delta_{ij}, & p+1\leq i\leq n\\ -\delta_{ij}, & 1\leq i\leq p\end{cases}$ (内积的负惯性指数为p)

故 $\iota_{e_1\wedge\cdots\wedge e_k}e^1\wedge\cdots\wedge e^n=(-1)^{\min\{k,p\}}e^{k+1}\wedge\cdots\wedge e^n$

再由 $*$ 算子的定义:

$$\begin{tikzcd}
\Lambda^k V \arrow[rr, "*"] \arrow[dr, "\iota"] & & \Lambda^{n-k}V \\
& \Lambda^{n-k}V^* \arrow[ur, "\text{duality}"'] &
\end{tikzcd}$$

$e_1\wedge\cdots\wedge e_k$ $\longmapsto$ $(-1)^{\min\{k,p\}}e_{k+1}\wedge\cdots\wedge e_n$

$(-1)^{\min\{k,p\}}e^{k+1}\wedge\cdots\wedge e^n$

这个映射依赖于 $\Lambda^n V$ 元素的选取!

从而：$*(e_1\wedge\cdots\wedge e_k)=(-1)^{\min\{k,p\}}e_{k+1}\wedge\cdots\wedge e_n$

类似地，可以计算：

$$(\iota_{e_{k+1}\wedge\cdots\wedge e_n}e^1\wedge\cdots\wedge e^n)(v_1,\cdots,v_k)$$

$$=e^1\wedge\cdots\wedge e^n(e_{k+1},\cdots,e_n,v_1,\cdots,v_k)$$

$$=(-1)^{k(n-k)}e^{k+1}\wedge\cdots\wedge e^n\wedge e^1\wedge\cdots\wedge e^k(e_{k+1},\cdots,e_n,v_1,\cdots,v_k)$$

$$=(-1)^{k(n-k)}(-1)^{\max\{0,p-k\}}e^1\wedge\cdots\wedge e^k(v_1,\cdots,v_k)$$

故 $*(e_{k+1}\wedge\cdots\wedge e_n)=(-1)^{k(n-k)}(-1)^{\max\{0,p-k\}}e_1\wedge\cdots\wedge e_k$

从而：$*^2(e_1\wedge\cdots\wedge e_k)=(-1)^{k(n-k)}(-1)^{\min\{p,k\}+\max\{0,p-k\}}e_1\wedge\cdots\wedge e_k$

$$=(-1)^{k(n-k)+p}e_1\wedge\cdots\wedge e_k.$$

因 $\tilde{e}_1\wedge\cdots\wedge\tilde{e}_n=e_1\wedge\cdots\wedge e_n$，故：

$$*^2(\tilde{e}_1\wedge\cdots\wedge\tilde{e}_k)=(-1)^{k(n-k)+p}\tilde{e}_1\wedge\cdots\wedge\tilde{e}_k$$

即 $*^2(e_{i_1}\wedge\cdots\wedge e_{i_k})=(-1)^{k(n-k)+p}e_{i_1}\wedge\cdots\wedge e_{i_k}$.

这就验证了 $*^2=(-1)^{k(n-k)+p}\mathrm{Id}$ #.

<u>Rmk</u>. Hodge $*$ 算子的另一等价形式可以表示为：指定 $\omega\in\wedge^n V^*$，$\alpha\in\wedge^k V$，$*\alpha\in\wedge^{n-k}V$ 满足：$\omega(\alpha\wedge*\alpha)=1$.

3. 直接计算即可.

<u>Rmk</u> 这里有一个小问题：对向量场的导数是什么？$\frac{\partial}{\partial s}X_{s,t}$，$\frac{\partial}{\partial t}Y_{s,t}$ 是什么？为避免问题，下面均认为这些都在局部坐标下存在.

$\forall p\in M$，取 p 附近的坐标卡 (f,U) 和 $\varphi_{s,t}(p)$ 附近的坐标卡 (g,V) 使得 $\varphi_{s,t}(U)\subset V$. 记 $x_0=f(p)$，x,y 分别为 (f,U)，(g,V) 的坐标.

$$\frac{\partial}{\partial t}\varphi_{s,t}(p)=\frac{\partial}{\partial t}(g\circ\varphi_{s,t}\circ f^{-1})(x_0)$$
$$=X_{s,t}\circ\varphi_{s,t}(p)$$
$$=(X^1_{s,t},\cdots,X^n_{s,t})\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)$$

类似地，$\frac{\partial}{\partial s}(g\circ\varphi_{s,t}\circ f^{-1})(x_0)$
$=Y_{s,t}\circ\varphi_{s,t}(p)$

两边同时对 s 求导数，则：

$$\frac{\partial}{\partial s}\left[(X^1_{s,t},\cdots,X^n_{s,t})\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)\right]$$
$$=\left(\frac{\partial}{\partial s}X^1_{s,t},\cdots,\frac{\partial}{\partial s}X^n_{s,t}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)$$
$$+\sum_{i=1}^n\Big(\left(\frac{\partial}{\partial y^i}X^1_{s,t}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x)\frac{\partial}{\partial s}(g\circ\varphi_{s,t}\circ f^{-1})^i(x_0),\cdots,$$
$$\left(\frac{\partial}{\partial y^i}X^n_{s,t}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x)\frac{\partial}{\partial s}(g\circ\varphi_{s,t}\circ f^{-1})^i(x_0)\Big)$$
$$=\left(\frac{\partial}{\partial s}X_{s,t}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)+\sum_{i=1}^n\left(Y^i_{s,t}\frac{\partial X^1_{s,t}}{\partial y^i},\cdots,Y^i_{s,t}\frac{\partial X^n_{s,t}}{\partial y^i}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)$$
$$=\frac{\partial^2}{\partial s\partial t}(g\circ\varphi_{s,t}\circ f^{-1})(x_0)$$

类似地，有：

$$\frac{\partial^2}{\partial t\partial s}(g\circ\varphi_{s,t}\circ f^{-1})(x_0)$$

$$=\left(\frac{\partial}{\partial t}Y_{s,t}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)$$

$$+\sum_{i=1}^{n}\left(X_{s,t}^{i}\frac{\partial Y_{s,t}^{1}}{\partial y^i},\cdots,X_{s,t}^{i}\frac{\partial Y_{s,t}^{n}}{\partial y^i}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0).$$

相减得到：

$$\left(\frac{\partial}{\partial s}X_{s,t}-\frac{\partial}{\partial t}Y_{s,t}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)$$

$$=\underbrace{\left(\frac{\partial^2}{\partial s\partial t}-\frac{\partial^2}{\partial t\partial s}\right)}_{}(g\circ\varphi_{s,t}\circ f^{-1})(x_0) \longrightarrow \text{光滑，故为}0$$

$$+\sum_{i=1}^{n}\left(X_{s,t}^{i}\frac{\partial Y_{s,t}^{1}}{\partial y^i}-Y_{s,t}^{i}\frac{\partial X_{s,t}^{1}}{\partial y^i},\cdots,X_{s,t}^{i}\frac{\partial Y_{s,t}^{n}}{\partial y^i}-Y_{s,t}^{i}\frac{\partial X_{s,t}^{n}}{\partial y^i}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)$$

$$=\sum_{i=1}^{n}\left(X_{s,t}^{i}\frac{\partial Y_{s,t}^{1}}{\partial y^i}-Y_{s,t}^{i}\frac{\partial X_{s,t}^{1}}{\partial y^i},\cdots,X_{s,t}^{i}\frac{\partial Y_{s,t}^{n}}{\partial y^i}-Y_{s,t}^{i}\frac{\partial X_{s,t}^{n}}{\partial y^i}\right)\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)$$

$$=\underbrace{\left(D_{X_{s,t}}Y_{s,t}^{1}-D_{Y_{s,t}}X_{s,t}^{1},\cdots,D_{X_{s,t}}Y_{s,t}^{n}-D_{Y_{s,t}}X_{s,t}^{n}\right)}_{\text{课上给出的Lie括号定义}}\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0)$$

$$=[X_{s,t},Y_{s,t}]\circ g\circ\varphi_{s,t}\circ f^{-1}(x_0),$$

课上给出的Lie括号定义.

从而：$\frac{\partial}{\partial s}X_{s,t}-\frac{\partial}{\partial t}Y_{s,t}=[X_{s,t},Y_{s,t}]$. #

<u>Rmk</u>. 课上给出了Lie括号的定义是在局部坐标下的

但实际上Lie括号的定义不依赖于坐标卡.

4. 证明 $L_X Y=[X,Y]$, $\forall X,Y\in\Gamma(TM)$.

下面利用 Cartan's Magic Formula 给出证明.

$\forall \omega\in\Omega^1(M)$, 先导出一个公式:

$$\omega(L_X Y)=X(\omega(Y))-(L_X\omega)(Y)$$

$$=X(\omega(Y))-(\iota_X d\omega)(Y)-(d\iota_X\omega)(Y)$$

$$=X(\omega(Y))-d\omega(X,Y)-d(\omega(X))(Y)$$

$$=X(\omega(Y))-Y(\omega(X))-d\omega(X,Y)=\omega([X,Y])$$

从而 $L_X Y=[X,Y]$. (因 $\Omega^1(M)=(TM)^*$) #

Rmk1 最直接的办法是在局部坐标下计算. 见 GTM218, P229.

Rmk2 注意 Cartan's Magic Formula 的证明是不需要 $L_X Y=[X,Y]$ 的

因此不会循环论证.

Rmk3 用到了 Leibniz 公式: 设 $\omega\in\Omega^1(M)$, $X,Y\in\Gamma(TM)$,

$$(L_X\omega)(Y)=X(\omega(Y))-\omega(L_X Y).$$

证明类似. (利用定义即可). 证明在下一页.

Rmk4 有同学问到了$[X,Y]$的定义.

Recall 课上证明了 $(L_X df)(Y) = Y(Xf)$

由此，将有：$\forall f \in C^\infty(M)$,

$$0 = d^2 f(Y,X) = Y(df(X)) - X(df(Y)) - df([X,Y]),$$

故 $[X,Y](f) = X(Yf) - Y(Xf)$, $\forall f \in C^\infty(M)$.

Pf of Rmk3:

$$(L_X\omega)(Y)_p = \lim_{t\to 0}\frac{(\varphi_t^X)^*_p\,\omega_{\varphi_t^X(p)}(Y_p) - \omega_p(Y_p)}{t}$$

$$= \lim_{t\to 0}\frac{\omega_{\varphi_t^X(p)}((d\varphi_t^X)_p(Y_p)) - \omega_p(Y_p)}{t}$$

$$= \lim_{t\to 0}\frac{\omega_{\varphi_t^X(p)}(Y_{\varphi_t^X(p)}) - \omega_p(Y_p)}{t} + \lim_{t\to 0}\frac{\omega_{\varphi_t^X(p)}((d\varphi_t^X)_p(Y_p) - Y_{\varphi_t^X(p)})}{t}$$

$$= \frac{(\omega(Y))_{\varphi_t^X(p)} - (\omega(Y))_p}{t}$$

$$\to X(\omega(Y))_p,\ t\to 0$$

$$= X(\omega(Y))_p + \lim_{t\to 0}\frac{(\varphi_t^X)^*_p\,\omega_{\varphi_t^X(p)}(Y_p - (d\varphi_{-t}^X)^*_{\varphi_t^X(p)}(Y_{\varphi_t^X(p)}))}{t}$$

$$Y_p - (d\varphi_{-t}^X)^*_{\varphi_t^X(p)}(Y_{\varphi_t^X(p)}) = -t(L_XY)_p + o(t)$$

Lie导数定义.

$$= X(\omega(Y))_p - \lim_{t\to 0}(\varphi_t^X)^*_p\,\omega_{\varphi_t^X(p)}((L_XY))_p - \underbrace{\lim_{t\to 0}\frac{(\varphi_t^X)^*_p\,\omega_{\varphi_t^X(p)}(o(t))}{t}}_{} = 0$$

$$(\varphi_t^X)^*_p\,\omega_{\varphi_t^X(p)} \xrightarrow{t\to 0} \omega$$

$$= X(\omega(Y))_p - \omega_p((L_XY)_p),\quad \forall p \in M$$

故 $(L_X\omega)(Y) = X(\omega(Y)) - \omega(L_XY)$. #

5. Poisson 括号满足 Jacobi 恒等式.

(有时候把 $[X,Y]$ 叫作 Lie 括号，本题定义的 $\{\cdot,\cdot\}$ 叫 Poisson 括号)

Recall 课上证明了：$L_{X_H}\omega = 0$ & $X_H(H) = 0$

$$\{\{H,G\},F\} = \omega(X_{\{H,G\}}, X_F) = -d\{H,G\}(X_F)$$

$$= -X_F(\{H,G\})$$

$$= -X_F(\omega(X_H, X_G))$$

(Leibniz) $$= -(L_{X_F}\omega)(X_H, X_G) - \omega(L_{X_F}X_H, X_G) - \omega(X_H, L_{X_F}X_G)$$

$$= -\omega([X_F, X_H], X_G) - \omega(X_H, [X_F, X_G])$$

类似地，$-\{\{G,F\},H\} = \omega([X_H, X_G], X_F) + \omega(X_G, [X_H, X_F])$

$$= \omega([X_H, X_G], X_F) + \omega([X_F, X_H], X_G),$$

$$-\{\{F,H\},G\} = \omega([X_G, X_F], X_H) + \omega(X_F, [X_G, X_H])$$

$$= \omega(X_H, [X_F, X_G]) + \omega([X_H, X_G], X_F),$$

故 $-(\{\{H,G\},F\} + \{\{G,F\},H\} + \{\{F,H\},G\})$

$$= 2\omega([X_F, X_H], X_G) + 2\omega(X_H, [X_F, X_G]) + 2\omega([X_H, X_G], X_F)$$

$$= 2[X_F, X_H](G) - 2[X_F, X_G](H) + 2[X_H, X_G](F)$$

$$= 2X_F(X_H(G)) - 2X_H(X_F(G)) - 2X_F(X_G(H))$$

$$+ 2X_G(X_F(H)) + 2X_H(X_G(F)) - 2X_G(X_H(F))$$

$d\omega=0$，故：

$$
\begin{aligned}
0 &= d\omega(X_H, X_G, X_F) \\
&= X_H(\omega(X_G, X_F)) - X_G(\omega(X_H, X_F)) + X_F(\omega(X_H, X_G)) \\
&\quad - \omega([X_H, X_G], X_F) + \omega([X_H, X_F], X_G) - \omega([X_G, X_F], X_H) \\
&= X_H(-X_F(G)) - X_G(-X_F(H)) + X_F(-X_G(H)) \\
&\quad -(-[X_H, X_G](F)) + (-[X_H, X_F](G)) - (-[X_G, X_F](H)) \\
&= X_G(X_F(H)) - X_F(X_G(H)) - X_H(X_F(G)) \\
&\quad + [X_H, X_G](F) - [X_H, X_F](G) + [X_G, X_F](H) \\
&= 2X_G(X_F(H)) - 2X_F(X_G(H)) - X_H(X_F(G)) + X_F(X_H(G)) \\
&\quad + X_H(X_G(F)) - X_G(X_H(F)) - X_H(X_F(G))
\end{aligned}
$$

且 $X_H(G) = dG(X_H) = -\omega(X_G, X_H) = \omega(X_H, X_G) = -dH(X_G) = -X_G(H)$

故上式 $\Longleftrightarrow 0 = 3X_G(X_F(H)) - 3X_F(X_G(H)) - 3X_H(X_F(G))$

$$\Longleftrightarrow X_G(X_F(H)) = X_F(X_G(H)) + X_H(X_F(G)) \quad (\star)$$

这样得到一组恒等式，下面代入计算.

$$-\left(\{\{H,G\},F\}+\{\{G,F\},H\}+\{\{F,H\},G\}\right)$$

$$=2X_F(X_H(G))-2X_H(X_F(G))-2X_F(X_G(H))$$

$$+2X_G(X_F(H))+2X_H(X_G(F))-2X_G(X_H(F))$$

$$=4X_F(X_H(G))-4X_H(X_F(G))-4X_G(X_H(F))$$

$$=4X_G(X_H(F))-4X_H(X_F(G))-4X_F(X_G(H))$$

$=0$，因（☆）成立. #

为什么？

Rmk. (M^{2n},ω) 称为辛流形 (Sympletic Manifolds)，$\dim M$ 为偶数.

Darboux定理：在 M 的任一点处，∃坐标卡 $((x,y),U)$ 使得 $\mathbb{R}^n\times\mathbb{R}^n\simeq\mathbb{R}^{2n}$.

$$\omega=\sum_{i=1}^n dx^i\wedge dy^i$$

可以算出 X_H：在上述的坐标卡下，有

$$X_H=\sum_{i=1}^n\left(-\frac{\partial H}{\partial y^i}\frac{\partial}{\partial x^i}+\frac{\partial H}{\partial x^i}\frac{\partial}{\partial y^i}\right),$$

这样，本题定义的 $\{\cdot,\cdot\}$ 可以表示为：

$$\{H,G\}=\omega(X_H,X_G)=X_H(G)$$

$$=\sum_{i=1}^n\left(\frac{\partial H}{\partial x^i}\frac{\partial G}{\partial y^i}-\frac{\partial H}{\partial y^i}\frac{\partial G}{\partial x^i}\right).$$

6. GTM218, Ex 8.10.

$\varphi(x,y)=(xy,\frac{y}{x})$. 不妨记坐标卡为 $\varphi:(x,y)\mapsto(u,v)=(\varphi^1,\varphi^2)$

$$J\varphi=\begin{pmatrix}\frac{\partial\varphi^1}{\partial x} & \frac{\partial\varphi^1}{\partial y}\\ \frac{\partial\varphi^2}{\partial x} & \frac{\partial\varphi^2}{\partial y}\end{pmatrix}=\begin{pmatrix}y & x\\ -\frac{y}{x^2} & \frac{1}{x}\end{pmatrix},\quad (x,y)\in\mathbb{R}_+^2$$

进而 $\varphi_*X=d\varphi(X)=x\left(\frac{\partial\varphi^1}{\partial x}\frac{\partial}{\partial u}+\frac{\partial\varphi^2}{\partial x}\frac{\partial}{\partial v}\right)+y\left(\frac{\partial\varphi^1}{\partial y}\frac{\partial}{\partial u}+\frac{\partial\varphi^2}{\partial y}\frac{\partial}{\partial v}\right)$

$$=xy\frac{\partial}{\partial u}-\frac{y}{x}\frac{\partial}{\partial v}+xy\frac{\partial}{\partial u}+\frac{y}{x}\frac{\partial}{\partial v}$$

$$=2xy\frac{\partial}{\partial u}=2u\frac{\partial}{\partial u},$$

$$\varphi_*Y=d\varphi(Y)=y\left(\frac{\partial\varphi^1}{\partial x}\frac{\partial}{\partial u}+\frac{\partial\varphi^2}{\partial x}\frac{\partial}{\partial v}\right)$$

$$=y^2\frac{\partial}{\partial u}-\frac{y^2}{x^2}\frac{\partial}{\partial v}$$

$$=uv\frac{\partial}{\partial u}-v^2\frac{\partial}{\partial v}.$$

#

7. GTM218, Ex 14.7(a)

$\varphi:(x,y)\mapsto(u,v)=(\varphi^1,\varphi^2)=(xy,e^{-y})$

在 (u,v) 坐标下，$\alpha=u\,dv$，于是

$$\varphi^*\alpha=u\,\varphi^*dv=xy\cdot(-e^{-y}dy)=-xye^{-y}dy.$$

知：$\varphi^*(d\alpha)=\varphi^*(du\wedge dv)=\varphi^*(du)\wedge\varphi^*(dv)=-ye^{-y}dx\wedge dy$,

$d(\varphi^*\alpha)=d(-xye^{-y}dy)=-ye^{-y}dx\wedge dy=\varphi^*(d\alpha)$.

#

8. GTM 218, Ex 9.17.

(1) 这结论需要用到ODE. 因 $X(p)\neq 0$, 取p处的一个坐标卡 $(\tilde{\varphi}, U)$.

不妨设 $\tilde{\varphi}(p)=0$, $X(p)=\frac{\partial}{\partial x^1}\Big|_0$. 适当缩小U, 使得:

$$X|_U := \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i},\quad f^1|_U \neq 0.$$

给定初值 $(u^2,\cdots,u^n)$, 解ODE问题:

$$\begin{cases} \frac{d}{dt} y^i(t;u^2,\cdots,u^n) = \frac{f^i(t,y^2,\cdots,y^n)}{f^1(t,y^2,\cdots,y^n)}, & i\geq 2 \\ y^i(0;u^2,\cdots,u^n)=u^i. \end{cases}$$

解的存在性,对初值光滑依赖性由ODE理论得到, 定义 φ:

$$\begin{cases} x^1 = u^1, \\ x^i = f^i(u^1;u^2,\cdots,u^n), \quad i\geq 2, \end{cases}$$

则 $J\varphi|_0 = Id$, 故 φ 为局部的微分同胚. 缩小U使得 $\varphi: V\to U$ 为微分同胚. 于是:

$$\begin{aligned}(f^1\circ\varphi)\varphi_*\left(\frac{\partial}{\partial u^1}\right) &= (f^1\circ\varphi)\sum_{i=1}^{n}\frac{\partial x^i}{\partial u^1}\left(\frac{\partial}{\partial x^i}\circ\varphi\right) \\ &= (f^1\circ\varphi)\left(\frac{\partial}{\partial u^1}\circ\varphi\right) + (f^1\circ\varphi)\sum_{i=2}^{n}\frac{f^i\circ\varphi}{f^1\circ\varphi}\left(\frac{\partial}{\partial x^i}\circ\varphi\right) \\ &= X\circ\varphi,\end{aligned}$$

这距离想要的结果只差一步.

再取微分同胚 ψ：

$$\begin{cases} v^1 = \int_0^{u^1} \frac{dt}{f'\circ\varphi(t,u^2,\cdots,u^n)} \\ v^i = u^i, \quad i \geq 2 \end{cases}$$

可以取 $\psi: V \to W$ 为微分同胚. 于是

$$\psi_*^{-1}\left(\frac{\partial}{\partial v^1}\right) = \sum_{i=1}^{n} \frac{\partial u^i}{\partial v^1}\left(\frac{\partial}{\partial u^i}\circ\psi^{-1}\right)$$

$$= \frac{\partial u^1}{\partial v^1}\left(\frac{\partial}{\partial u^1}\circ\psi^{-1}\right) = (f'\circ\varphi\circ\psi^{-1})\left(\frac{\partial}{\partial u^1}\circ\psi^{-1}\right)$$

$$= \varphi_*^{-1}(X)\circ\varphi\circ\psi^{-1},$$

从而在这个坐标卡下, $X = \frac{\partial}{\partial v^1}$.

(2) <u>Check</u> $[X_1, X_2] = -X_3$, $[X_1, X_3] = X_2$, $[X_2, X_3] = -X_1$.

注意到: $zV_1 + xV_2 = \cancel{xz\frac{\partial}{\partial y}} - yz\frac{\partial}{\partial x} + xy\frac{\partial}{\partial z} - \cancel{xz\frac{\partial}{\partial y}}$

$$= y\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right) = -yV_3$$

这表明 $\{X_1, X_2, X_3\}$ 是线性相关的 (在 $(1,0,0)$ 附近)

因此不存在坐标卡使得 $X_i = \partial/\partial x^i$.

9. Killing 向量场.

首先导出 X 满足的条件. 记 $\mathbb{R}^3$ 上的欧氏联络为 D, 则:

$$X(g(Y,Z)) = g(D_X Y, Z) + g(Y, D_X Z)$$

对 $\forall X, Y, Z \in \Gamma(TM)$. 于是:

$$(L_X g)(Y,Z) = 0 = X(g(Y,Z)) - g(L_X Y, Z) - g(Y, L_X Z)$$
$$= g(D_X Y - L_X Y, Z) + g(Y, D_X Z - L_X Z).$$

且 $D_X Y - D_Y X = [X,Y]$, $L_X Y = [X,Y]$, 故.

$$g(D_Y X, Z) + g(Y, D_Z X) = 0$$

这就是 Killing 向量场的另一等价定义.

在 $\mathbb{R}^3$ 的坐标下, 记 $X = X^i \partial_i$, 则:

$$g(D_{\partial_i} X, \partial_j) + g(\partial_i, D_{\partial_j} X) = 0$$
$$= g(\partial_i X^k \partial_k, \partial_j) + g(\partial_i, \partial_j X^k \partial_k)$$
$$= \partial_i X^k g_{jk} + \partial_j X^k g_{ik} = 0$$

记 $G = (g_{ij})_{n\times n}$, 则 $G = \mathrm{diag}(1,1,-1)$; 记 $A = (\partial_i X^j)_{n\times n}$,

则 $AG + GA^T = 0 = (AG) + (AG)^T$

展开写为：$\begin{pmatrix} \partial_1 X^1 & \partial_1 X^2 & -\partial_1 X^3 \\ \partial_2 X^1 & \partial_2 X^2 & -\partial_2 X^3 \\ \partial_3 X^1 & \partial_3 X^2 & -\partial_3 X^3 \end{pmatrix}$ 为反对称的.

故 $\left.\begin{array}{ll} \partial_1 X^1 = \partial_2 X^2 = \partial_3 X^3 = 0, & \\ \partial_1 X^2 + \partial_2 X^1 = 0 & ① \\ \partial_1 X^3 = \partial_3 X^1 & ② \\ \partial_2 X^3 = \partial_3 X^2 & ③ \end{array}\right\}$

设 $X^1 = f(y,z)$

$X^2 = h(x,z)$

$X^3 = k(x,y)$

求解方程组.

① $\Leftrightarrow \dfrac{\partial h}{\partial x} + \dfrac{\partial f}{\partial y} = 0 \Rightarrow \dfrac{\partial^2 h}{\partial x^2} = 0,\ \dfrac{\partial^2 f}{\partial y^2} = 0$

② $\Leftrightarrow \dfrac{\partial k}{\partial x} = \dfrac{\partial f}{\partial z} \Rightarrow \dfrac{\partial^2 k}{\partial x^2} = 0,\ \dfrac{\partial^2 f}{\partial z^2} = 0$

③ $\Leftrightarrow \dfrac{\partial h}{\partial z} = \dfrac{\partial k}{\partial y} \Rightarrow \dfrac{\partial^2 h}{\partial z^2} = 0,\ \dfrac{\partial^2 k}{\partial y^2} = 0$

以及：$\left.\begin{array}{l} \dfrac{\partial^2 f}{\partial y \partial z} = \dfrac{\partial^2 k}{\partial x \partial y} = \dfrac{\partial^2 h}{\partial x \partial z} \quad (②+③) \\ \dfrac{\partial^2 f}{\partial y \partial z} = -\dfrac{\partial^2 h}{\partial x \partial z} \quad (①) \end{array}\right\}$ 3项均为0.

从而 f, h, k 均为线性函数，$\begin{cases} f = ay + bz + d_1, \\ h = -ax + cz + d_2, \\ k = bx + cy + d_3, \end{cases}$ $\begin{array}{l} a, b, c, \\ d_1, d_2, d_3 \end{array} \in \mathbb{R}$

(1) 所有的Killing向量场均具有形式：

$$X=a(y,-x,0)+b(z,0,x)+c(0,z,y)+(d_1,d_2,d_3),$$

其中 $a,b,c,d_1,d_2,d_3\in\mathbb{R}$；

(2). 直接计算，分别记 $X_1=(y,-x,0)$，$X_2=(z,0,x)$，$X_3=(0,z,y)$，则：

$$\begin{aligned}
[X_1,X_2]&=y\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial x}+x\frac{\partial}{\partial z}\right)-x\frac{\partial}{\partial y}\left(z\frac{\partial}{\partial x}+x\frac{\partial}{\partial z}\right)\\
&\quad-z\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)-x\frac{\partial}{\partial z}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)\\
&=y\frac{\partial}{\partial z}+z\frac{\partial}{\partial y}=X_3,\\
[X_1,X_3]&=y\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}\right)-x\frac{\partial}{\partial y}\left(z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}\right)\\
&\quad-z\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)-y\frac{\partial}{\partial z}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)\\
&=-x\frac{\partial}{\partial z}-z\frac{\partial}{\partial x}=-X_2,\\
[X_2,X_3]&=z\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}\right)+x\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial y}+y\frac{\partial}{\partial y}\right)\\
&\quad-z\frac{\partial}{\partial y}\left(z\frac{\partial}{\partial x}+x\frac{\partial}{\partial z}\right)-y\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial x}+x\frac{\partial}{\partial z}\right)\\
&=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}=-X_1,
\end{aligned}$$

Lie括号运算在 $\mathrm{span}\{X_1,X_2,X_3\}$ 下是封闭的. #

因此：若 X,Y 为Killing场，则 $[X,Y]$ 也是Killing场

10. 切触形式 (Contact Form)

(1) 取 M 的一个坐标覆盖图册 $\{(\varphi_\beta, U_\beta)\}$, $\varphi_\beta : U_\beta \xrightarrow{diff} \mathbb{R}^n$.

给 TM 赋予一个 Riemann 度量 g, 定义 $A : \Gamma(TM) \to \Gamma(TM)$,

$$g(AX, Y) := d\alpha(X, Y), \ \forall X, Y \in \Gamma(TM),$$

实际上 $A \in \Gamma(T^{(1,1)}M)$. 且: $g(AX, Y) = -g(X, AY)$

实矩阵的相似理论 $\Rightarrow$ $A|_{U_\beta}$ 有唯一的一组 0 特征子空间,

$\alpha \wedge d\alpha$ 处处不消失 $\Rightarrow$ $d\alpha$ 处处不为 0

故 A 处处不恒为 0

取 $A|_{U_\beta}$ 的一个 0 特征方向 $R_\beta \in \Gamma(TU_\beta)$ 满足 $\alpha|_{U_\beta}(R_\beta) = 1$

Fact α 在 A 的 0 特征子空间上不为 0.

若不然, 取 A 的一个 0 特征方向 $R \neq 0$, $\alpha(R) = 0$, 则:

$$\iota_R(\alpha \wedge d\alpha) = (\iota_R \alpha) \wedge d\alpha + \alpha \wedge (\iota_R d\alpha)$$

$$= \alpha(R) d\alpha + \alpha \wedge (\iota_R d\alpha) = 0$$

这与 $\alpha \wedge d\alpha$ 处处不消失矛盾!

定义 $R_\alpha \in \Gamma(TM)$: $R_\alpha|_{U_\beta} = R_\beta$

Check 定义合理.

若 $U_\beta \cap U_\gamma \neq \emptyset$，在 $(\varphi_\beta, U_\beta \cap U_\gamma)$ 上，设 $R_\gamma = \lambda_\gamma^\beta R_\beta$

则：$\alpha|_{U_\beta}(R_\gamma) = 1 = \alpha|_{U_\beta}(\lambda_\gamma^\beta R_\beta) = \lambda_\gamma^\beta$

故 $R_\gamma = R_\beta$. 从而定义合理.

由此定义的 R_α 就满足 $\iota_{R_\alpha} d\alpha = 0$ 且 $\alpha(R_\alpha) = 1$.

(2) $\iota_{R_\alpha} d\alpha = 0$. $\iota_{R_\alpha}\alpha = \alpha(R_\alpha) = 1$. 故：

$$L_{R_\alpha}\alpha = d(\iota_{R_\alpha}\alpha) + \iota_{R_\alpha} d\alpha = 0.$$

(3) $\mathbb{R}^3$ 上的一个切触形式 $\alpha = dx + y\,dz$

Check：$\alpha \wedge d\alpha = dx \wedge dy \wedge dz$ 处处不消失

$(d\alpha = dy \wedge dz)$

对应的 $R_\alpha = \frac{\partial}{\partial x}$. $\alpha(R_\alpha) = dx(R_\alpha) + y\,dz(R_\alpha) = 1$,

$$\iota_{R_\alpha} d\alpha = \iota_{\frac{\partial}{\partial x}} dy \wedge dz = 0.$$

Rmk 实际上 M 是可定向的，因 $\alpha \wedge d\alpha$ 处处不消失.

见 GTM218：$\exists\, \omega \in \Omega^n(M)$, $n = \dim(M)$, ω 处处不消失

$\Longleftrightarrow M$ 可定向.